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# Asymptotic properties of the maximum likelihood and cross validation estimators for transformed Gaussian processes

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**Abstract:** The asymptotic analysis of covariance parameter estimation of Gaussian processes has been subject to intensive investigation. However, this asymptotic analysis is very scarce for non-Gaussian processes. In this paper, we study a class of non-Gaussian processes obtained by regular non-linear transformations of Gaussian processes. We provide the increasing-domain asymptotic properties of the (Gaussian) maximum likelihood and cross validation estimators of the covariance parameters of a non-Gaussian process of this class. We show that these estimators are consistent and asymptotically normal, although they are defined as if the process was Gaussian. They do not need to model or estimate the non-linear transformation. Our results can thus be interpreted as a robustness of (Gaussian) maximum likelihood and cross validation towards non-Gaussianity. Our proofs rely on two technical results that are of independent interest for the increasing-domain asymptotic literature of spatial processes. First, we show that, under mild assumptions, coefficients of inverses of large covariance matrices decay at an inverse polynomial rate as a function of the corresponding observation location distances. Second, we provide a general central limit theorem for quadratic forms obtained from transformed Gaussian processes. Finally, our asymptotic results are illustrated by numerical simulations.

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## 1. Introduction

Kriging [48, 40] consists of inferring the values of a (Gaussian) random field given observations at a finite set of points. It has become a popular method for a large range of applications, such as geostatistics [37], numerical code approximation [41, 43, 9], calibration [39, 10], global optimization [33], and machine learning [40].

When considering a Gaussian process, one has to deal with the estimation of its covariance function. Usually, it is assumed that the covariance function belongs to a given parametric family (see [1] for a review of classical families). In this case, the estimation boils down to estimating the corresponding covariance parameters. Nowadays, the main estimation techniques are based on maximum likelihood [48, 40], cross-validation [59, 6, 7, 14] and variation estimators [31, 3, 4].

The asymptotic properties of estimators of the covariance parameters have been widely studied in the two following frameworks. The fixed-domain asymptotic framework, sometimes called infill asymptotics [48, 21], corresponds to the case where more and more data are observed in some fixed bounded sampling domain. The increasing-domain asymptotic framework corresponds to the case where the sampling domain increases with the number of observed data.

Under fixed-domain asymptotics, and particularly in low dimensional settings, not all covariance parameters can be estimated consistently (see [30, 48]). Hence, the distinction is made between microergodic and non-microergodic covariance parameters [30, 48]. Although non-microergodic parameters cannot be estimated consistently, they have an asymptotically negligible impact on prediction [45, 46, 47, 58]. There is, however, a fair amount of literature on the consistent estimation of microergodic parameters (see for instance [58, 34, 24, 51, 56, 57]).

This paper focuses on the increasing-domain asymptotic framework. Indeed, generally speaking, increasing-domain asymptotic results hold for significantly more general families of covariance functions than fixed-domain ones. Under increasing-domain asymptotics, the maximum likelihood and cross validation estimators of the covariance parameters are consistent and asymptotically normal under mild regularity conditions [36, 44, 7, 28].

All the asymptotic results discussed above are based on the assumption that the data come from a Gaussian random field. This assumption is indeed theoretically convenient but might be unrealistic for real applications. When the data stem from a non-Gaussian random field, it is still relevant to estimate the covariance function of this random field. Hence, it would be valuable to extend the asymptotic results discussed above to the problem of estimating the covariance parameters of a non-Gaussian random field.

In this paper, we provide such an extension, in the special case where the non-Gaussian random field is a deterministic (unknown) transformation of a Gaussian random field. Models of transformed Gaussian random fields have been used extensively in practice (for example in [20, 53, 2, 55]).

We provide various asymptotic results, under reasonable regularity assumptions. In particular, our assumptions on the transformation function are mild. For most of our results, only sub-exponentiality is required.

We prove that applying the (Gaussian) maximum likelihood estimator to data from a transformed Gaussian random field yields a consistent and asymptotically normal estimator of the covariance parameters of the transformed random field. This (Gaussian) maximum likelihood estimator corresponds to what would typically be done in practice when applying a Gaussian process model to a non-Gaussian spatial process. This estimator does not need to know the existence of the non-linear transformation function and is not based on the exact density of the non-Gaussian data. We refer to Remark 5 for further details and discussion on this point.

We then obtain the same consistency and asymptotic normality result when considering a cross validation estimator. In addition, we establish the joint asymptotic normality of both these estimators, which provides the asymptotic distribution of a large family of aggregated estimators. Our asymptotic results on maximum likelihood and cross validation are illustrated by numerical simulations.

To the best of our knowledge, our results (Theorems 3, 4, 5, 6 and 7) provide the first increasing-domain asymptotic analysis of Gaussian maximum likelihood and cross validation for non-Gaussian random fields. Our proofs intensively rely on Theorems 1 and 2. Theorem 1 shows that the components of inverse covariance matrices are bounded by inverse polynomial functions of the corresponding distance between observation locations. Theorem 2 provides a generic central limit theorem for quadratic forms constructed from transformed Gaussian processes. These two theorems have an interest in themselves.

The rest of the paper is organized as follows. In Section 2, general properties of transformed Gaussian processes are provided. In Section 3, Theorems 1 and 2 are stated. In Section 4, an application of these two theorems is given to the case of estimating a single variance parameter. In Section 5, the consistency and asymptotic normality results for general covariance parameters are given. The joint asymptotic normality result is also given in this section. Section 6 discusses some extensions of the results of the previous sections. The simulation results are provided in Section 7. All the proofs are provided in the appendix.

## 2. General properties of transformed Gaussian processes

In applications, the use of Gaussian process models may be too restrictive. One possibility for obtaining larger and more flexible classes of random fields is to consider transformations of Gaussian processes. In this section, we now introduce the family of transformed Gaussian processes that we will study asymptotically in this paper. This family is determined by regularity conditions on the covariance function of the original Gaussian process and on the transformation function.

Let us first introduce some notation. Throughout the paper,  $C_{\inf} > 0$  (resp.  $C_{\sup} < \infty$ ) denotes a generic strictly positive (resp. finite) constant. This constant never depends on the number of observations  $n$ , or on the covariance parameters (see Section 5), but is allowed to depend on other variables. We mention these dependences explicitly in cases of ambiguity. The values of  $C_{\inf}$  and  $C_{\sup}$  may change across different occurrences.

For a vector  $x$  of dimension  $d$  we let  $|x| = \max_{i=1,\dots,d} |x_i|$ . Further, the Euclidean and operator norms are denoted by  $\|x\|$  and by  $\|M\|_{op} = \sup\{\|Mx\| : \|x\| \leq 1\}$ , for any matrix  $M$ . We let  $\lambda_1(B) \geq \dots \geq \lambda_r(B)$  be the  $r$  eigenvalues of a  $r \times r$  symmetric matrix  $B$ . We let  $\rho_1(B) \geq \dots \geq \rho_r(B) \geq 0$  be the  $r$  singular values of a  $r \times r$  matrix  $B$ . We let  $\mathbb{N}$  be the set of non-zero natural numbers.

Further, we define the Fourier transform of a function  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $\hat{h}(f) = (2\pi)^{-d} \int_{\mathbb{R}^d} h(t) e^{-if^\top t} dt$ , where  $i^2 = -1$ , for  $f \in \mathbb{R}^d$ . When mentioning the Fourier transform of a function  $h$ , we implicitly assume that Fourier inversion holds, that is  $\hat{h}$  is summable and, for  $s \in \mathbb{R}^d$ ,  $h(s) = \int_{\mathbb{R}^d} \hat{h}(f) e^{is^\top f} df$ .

For a sequence of observation locations, the next condition ensures that a fixed distance between any two observation locations exists. This condition is classical [7, 12].

**Condition 1.** We say that a sequence of observation locations,  $(x_i)_{i \in \mathbb{N}}$ ,  $x_i \in \mathbb{R}^d$ , is asymptotically well-separated if we have  $\inf_{i,j \in \mathbb{N}, i \neq j} |x_i - x_j| > 0$ .

The next condition on a stationary covariance function is classical under increasing-domain asymptotics. This condition provides asymptotic decorrelation for pairs of distant observation locations and implies that covariance matrices are asymptotically well-conditioned when a minimal distance between any two distinct observation locations exists [36, 7].

**Condition 2.** We say that a stationary covariance function  $k$  on  $\mathbb{R}^d$  is sub-exponential and asymptotically positive if:

i)  $C_{\sup}$  and  $C_{\inf}$  exist such that, for all  $s \in \mathbb{R}^d$ , we have

$$|k(s)| \leq C_{\sup} \exp(-C_{\inf}|s|); \quad (1)$$

ii) For any sequence  $(x_i)_{i \in \mathbb{N}}$  satisfying Condition 1, we have  $\inf_{n \in \mathbb{N}} \lambda_n(\Sigma) > 0$ , where  $\Sigma$  is the  $n \times n$  matrix  $(k(x_i - x_j))_{i,j=1,\dots,n}$ .

In Condition 2, we remark that  $k : \mathbb{R}^d \rightarrow \mathbb{R}$  is called a stationary covariance function in the sense that  $(x_1, x_2) \rightarrow k(x_1 - x_2)$  is a covariance function. We use this slight language abuse for convenience.

We also remark that, when non-transformed Gaussian processes are considered, a polynomial decay of the covariance function in Condition 2 i) is sufficient to obtain asymptotic results [7, 8]. Here an exponential decay is needed in the proofs to deal with the non-Gaussian case. Nevertheless, most classical covariance functions satisfy inequality (1). In particular, the Matérn covariance functions [48], the generalized Wendland covariance functions [17] and some of the power exponential covariance functions [43] satisfy inequality (1). On the

other hand, for instance, the covariance functions in the Cauchy class [29] do not satisfy inequality (1).

When considering a transformed Gaussian process, we will consider a transformation satisfying the following regularity condition, which enables us to subsequently obtain regularity conditions on the covariance function of the transformed Gaussian process.

**Condition 3.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a fixed non-constant continuously differentiable function, with derivative  $F'$ . We say that  $F$  is sub-exponential and non-decreasing if:*

- i) For all  $t \in \mathbb{R}$ , we have  $|F(t)| \leq C_{\text{sup}} \exp(C_{\text{sup}}|t|)$  and  $|F'(t)| \leq C_{\text{sup}} \exp(C_{\text{sup}}|t|)$ ;*
- ii) The function  $F$  is non-decreasing on  $\mathbb{R}$ .*

Regarding Condition 3 ii), we point out that many transformations of Gaussian processes considered in the literature are indeed non-decreasing, for instance the Tukey's  $g$ -and- $h$  transformation in [53] and the exponential transformation for log-Gaussian processes. Furthermore, Condition 3 ii) need not always be assumed for the results of the present paper to hold, see Remarks 4 and 9.

In the following lemma, we show that the covariance function of a transformed Gaussian process satisfies Condition 2, when Conditions 2 and 3 are satisfied, for the original process and for the transformation.

**Lemma 1.** *Assume that the stationary covariance function  $k$  satisfies Condition 2 and that the transformation  $F$  satisfies Condition 3. Let  $X$  be a zero-mean Gaussian process with covariance function  $k$  and let  $k'$  be the stationary covariance function of  $F(X(\cdot))$ . Then,  $k'$  satisfies Condition 2.*

In the next lemma, we show that we can replace the condition of an increasing-transformation by the condition of a monomial transformation of even degree (with an additive constant).

**Lemma 2.** *If a covariance function  $k$  satisfies Condition 2 i) and if the Fourier transform  $\hat{k}$  of  $k$  is strictly positive on  $\mathbb{R}^d$ , then  $k$  satisfies Condition 2 ii). Furthermore, in this case, in Lemma 1, Condition 3 ii) can be replaced by the condition  $F(x) = x^{2r} + u$  for  $r \in \mathbb{N}$ ,  $u \in \mathbb{R}$  and  $x \in \mathbb{R}$ .*

### 3. Two main technical results

#### 3.1. Transformed Gaussian process framework

In the rest of the paper, we will consider an unobserved Gaussian process  $Z$  on  $\mathbb{R}^d$  with  $d \in \mathbb{N}$  fixed. Assume that  $Z$  has zero-mean and stationary covariance function  $k_Z$ . We assume throughout that  $k_Z$  satisfies Condition 2.

We consider a fixed transformation function  $T$  satisfying Condition 3. We assume that we observe the transformed Gaussian process  $Y$ , defined by  $Y(s) = T(Z(s))$  for any  $s \in \mathbb{R}^d$ .

We assume throughout that the random field  $Y$  has zero-mean. We remark that, for a non-linear transformation  $F : \mathbb{R} \rightarrow \mathbb{R}$ , for  $s \in \mathbb{R}^d$ , the fact that  $Z(s)$  has zero-mean does not imply that the random variable  $F(Z(s))$  has zero-mean. Hence, we implicitly assume that  $T$  is of the form  $F - \mathbb{E}[F(Z(x))]$ , where  $F$  satisfies Condition 3 and  $x \in \mathbb{R}^d$  is arbitrary. Note that  $\mathbb{E}[F(Z(x))]$  is constant in  $x$  by stationarity and that, if  $F$  satisfies Condition 3 or the condition specified in Lemma 2, then  $F - \mathbb{E}[F(Z(x))]$  also satisfies these conditions. We remark that the assumption of zero-mean (or equivalently of a known mean function for  $Y$ ) is common in the literature. We provide a further discussion of this assumption in Section 6.2.

We let  $k_Y$  be the covariance function of  $Y$ . We remark that, from Lemma 1 (applied with  $F = T$  and  $X = Z$  and thus  $k' = k_Y$ ),  $k_Y$  also satisfies Condition 2.

We let  $(s_i)_{i \in \mathbb{N}}$  be the sequence of observation locations, with  $s_i \in \mathbb{R}^d$  for  $i \in \mathbb{N}$ . We assume that  $(s_i)_{i \in \mathbb{N}}$  satisfies Condition 1.

For  $n \in \mathbb{N}$ , we let  $y = (y_1, \dots, y_n)^\top = (Y(s_1), \dots, Y(s_n))^\top$  be the (non-Gaussian) observation vector and  $R = (k_Y(s_i - s_j))_{i,j=1,\dots,n}$  be its covariance matrix.

The problem of estimating the covariance function  $k_Y$  from the observation vector  $y$  is crucial and has been extensively studied in the Gaussian case (when  $T$  is a linear function). Classically, we assume that  $k_Y$  belongs to a parametric family of covariance functions. We will provide the asymptotic properties of two of the most popular estimators of the covariance parameters: the one based on the (Gaussian) maximum likelihood [40, 48] and the one based on cross validation [14, 6, 59]. To our knowledge, such properties are currently known only for Gaussian processes, and we will provide analogous properties in the transformed Gaussian framework.

### 3.2. Bounds on the elements of inverse covariance matrices

In the case of (non-transformed) Gaussian processes, one important argument for establishing the asymptotic properties of the maximum likelihood and cross validation estimators is to bound the largest eigenvalue of the inverse covariance matrix  $R^{-1}$ . Unfortunately, due to the non-linearity of the transformation  $T$ , such a bound on the largest eigenvalue is no longer sufficient in our setting.

To circumvent this issue, we obtain in the following theorem stronger control over the matrix  $R^{-1}$ : we show that its coefficients decrease polynomially quickly with respect to the corresponding distance between observation locations. This theorem may have an interest in itself.

**Theorem 1.** *Consider the setting of Section 3.1. For all fixed  $0 < \tau < \infty$ , we have, for all  $n \in \mathbb{N}$  and  $i, j = 1, \dots, n$*

$$\left| (R^{-1})_{i,j} \right| \leq \frac{C_{\sup}}{1 + |s_i - s_j|^{d+\tau}},$$

where  $C_{\sup}$  depends on  $\tau$  but does not depend on  $n, i, j$ .

### 3.3. Central limit theorem for quadratic forms of transformed Gaussian processes

In the proofs on covariance parameter estimation of Gaussian processes, a central step is to show the asymptotic normality of quadratic forms of large Gaussian vectors. This asymptotic normality is established by diagonalizing the matrices of the quadratic forms. This diagonalization provides sums of squares of decorrelated Gaussian variables and thus sums of independent variables [31, 7].

In the transformed Gaussian case, one has to deal with quadratic forms involving transformations of Gaussian vectors. Hence, the previous arguments are not longer valid. To overcome this issue, we provide below a general central limit theorem for quadratic forms of transformed Gaussian vectors. This theorem may have an interest in itself.

This asymptotic normality result is established by considering a metric  $d_w$  generating the topology of weak convergence on the set of Borel probability measures on Euclidean spaces (see, e.g., [26] p. 393). We prove that the distance between the sequence of the standardized distributions of the quadratic forms and Gaussian distributions decreases to zero when  $n$  increases. The introduction of the metric  $d_w$  enables us to formulate asymptotic normality results in cases when the sequence of standardized variances of the quadratic forms does not necessarily converge as  $n \rightarrow \infty$ .

**Theorem 2.** *Consider the setting of Section 3.1. Let  $(A_n)_{n \in \mathbb{N}}$  be a sequence of matrices such that  $A_n$  has dimension  $n \times n$  for any  $n \in \mathbb{N}$ . Let  $A = A_n$  for concision. Assume that for all  $n \in \mathbb{N}$  and for all  $i, j = 1, \dots, n$ ,*

$$|A_{i,j}| \leq \frac{C_{\sup}}{1 + |s_i - s_j|^{d+C_{\inf}}},$$

where  $C_{\sup}$  does not depend on  $i, j$ . Let

$$V_n = \frac{1}{n} y^\top A y. \quad (2)$$

Let  $\mathcal{L}_n$  be the distribution of  $\sqrt{n}(V_n - \mathbb{E}[V_n])$ . Then, as  $n \rightarrow \infty$ ,

$$d_w(\mathcal{L}_n, \mathcal{N}[0, n \operatorname{Var}(V_n)]) \rightarrow 0.$$

In addition, the sequence  $(n \operatorname{Var}(V_n))_{n \in \mathbb{N}}$  is bounded.

**Remark 1.** *In the case where limits  $E_\infty$  and  $\sigma_\infty$  exist, such that*

$$\mathbb{E}[V_n] - E_\infty = o(n^{-1/2})$$

and the sequence  $(n \operatorname{Var}(V_n))_{n \in \mathbb{N}}$  converges to a fixed variance  $\sigma_\infty^2$ , the result of Theorem 2 can be written in the classical form

$$\sqrt{n}(V_n - E_\infty) \xrightarrow{\mathcal{L}} \mathcal{N}[0, \sigma_\infty^2],$$

as  $n \rightarrow \infty$ .



**Remark 2.** Throughout Section 3, we assume that  $k_Z$  satisfies Condition 2. Note that this condition does not impose  $k_Z$  to be continuous at zero. Hence, Theorems 1 and 2 allow for a decomposition  $k_Z(s) = \bar{k}_Z(s) + \eta 1_{s=0}$ , for  $s \in \mathbb{R}^d$ , where  $\bar{k}_Z : \mathbb{R}^d \rightarrow \mathbb{R}$  is a continuous stationary covariance function and  $\eta \geq 0$ . Hence, Theorems 1 and 2 allow for an additive nugget effect, with variance  $\eta$ , on the unobserved Gaussian process  $Z$ . Furthermore, we remark that in this case, when  $\eta > 0$ ,  $k_Z$  automatically satisfies Condition 2 ii), as the eigenvalues of covariance matrices obtained from  $k_Z$  are larger than  $\eta$ .

**Remark 3.** One can check that in the proofs of Theorems 1 and 2, it is not necessary that  $k_Z$  satisfies Condition 2 ii). Hence, these two theorems hold whenever  $T$  satisfies Condition 3,  $k_Y$  satisfies Condition 2 i) and ii) and  $k_Z$  satisfies Condition 2 i). There are various assumptions that can be made, in order to guarantee that  $k_Y$  satisfies Condition 2 ii).

First, if a nugget effect as described in Remark 2 is assumed on the covariance function  $k_Z$ , then  $k_Z$  automatically satisfies Condition 2 ii) and thus, from Lemma 1,  $k_Y$  satisfies Condition 2 ii).

Second, it would be possible to assume that for  $s \in \mathbb{R}^d$ , we have  $Y(s) = T(Z(s)) + \tilde{N}(s)$ , where  $Y$  and  $Z$  still have zero-mean and where  $\tilde{N}$  is a centered stationary random field on  $\mathbb{R}^d$ , independent of  $Z$ , with  $\text{Cov}(\tilde{N}(u), \tilde{N}(v)) = \eta 1_{u=v}$ , for  $u, v \in \mathbb{R}^d$ , with  $\eta > 0$ . In this case, the covariance function  $k_Y$  of  $Y$  satisfies Condition 2 ii), similarly as in Remark 2. Furthermore, consider the case where there exists  $\tilde{T} : \mathbb{R} \rightarrow \mathbb{R}$  satisfying Condition 3, such that, for  $s \in \mathbb{R}^d$ ,  $\tilde{N}(s) = \tilde{T}(\zeta(s))$ , where  $\zeta$  is a centered Gaussian random field, independent of  $Z$ , with  $\text{Cov}(\zeta(u), \zeta(v)) = \kappa 1_{u=v}$  for  $u, v \in \mathbb{R}^d$ , with  $\kappa > 0$ . In this case, one can check that Theorems 1 and 2 hold in this modified setting for  $Y$ , with proofs that are minor but straightforward modifications of those given in the appendix (see also the proof of Proposition 1 in the appendix). Note that this modified setting is equivalent to letting the observation vector  $y$  be defined by  $y_i = T(Z(s_i)) + \xi_i$ ,  $i = 1, \dots, n$ , where  $\xi_1, \dots, \xi_n$  are i.i.d., independent of  $Z$ , with zero-mean and variance  $\eta$ , since  $s_1, \dots, s_n$  are two-by-two distinct.

Third, in the case where  $k_Y$  and  $k_Z$  are continuous, we have shown in Lemma 1 that if  $k_Z$  satisfies Condition 2 ii), then  $k_Y$  satisfies Condition 2 ii). In Sections 3 and 4, we thus simply assume that  $k_Z$  satisfies Condition 2 ii). A classical assumption that guarantees  $k_Z$  to satisfy Condition 2 ii) is that  $k_Z$  has a strictly positive Fourier transform [7, 12] (see also the proof of Lemma 2).

Finally, when  $k_Y$  is continuous, one could also assume that its Fourier transform is strictly positive to guarantee that  $k_Y$  satisfies Condition 2 ii). In fact, this is what we do in Section 5, see Condition 5.

#### 4. Estimation of a single variance parameter

We let  $\sigma_0^2$  be the marginal variance of  $Y$ , that is  $\text{Var}(Y(s)) = \sigma_0^2$  for any  $s \in \mathbb{R}^d$ . We let  $k_Y = \sigma_0^2 c_Y$  be the stationary covariance function of  $Y$ , where  $c_Y$  is a correlation function. We assume that the same conditions as in Section 3 hold. Then, the standard Gaussian maximum likelihood estimator of the variance

parameter is

$$\hat{\sigma}_{\text{ML}}^2 = \frac{1}{n} y^\top C^{-1} y,$$

where  $C = (c_Y(s_i - s_j))_{1 \leq i, j \leq n}$ . One can simply show that  $\mathbb{E}[\hat{\sigma}_{\text{ML}}^2] = \sigma_0^2$  even though  $y$  is not a Gaussian vector, since  $y$  has mean vector 0 and covariance matrix  $\sigma_0^2 C$ . Hence, a direct consequence of Theorems 1 and 2 is then that the maximum likelihood estimator is asymptotically Gaussian, with a  $n^{1/2}$  rate of convergence, even though the transformed process  $Y$  is not a Gaussian process.

**Corollary 1.** *Let  $\mathcal{L}_n$  be the distribution of  $\sqrt{n}(\hat{\sigma}_{\text{ML}}^2 - \sigma_0^2)$ . Then, as  $n \rightarrow \infty$ ,*

$$d_w(\mathcal{L}_n, \mathcal{N}[0, n \text{Var}(\hat{\sigma}_{\text{ML}}^2 - \sigma_0^2)]) \rightarrow 0.$$

*In addition, the sequence  $(n \text{Var}(\hat{\sigma}_{\text{ML}}^2 - \sigma_0^2))_{n \in \mathbb{N}}$  is bounded.*

The proof of Theorem 2 actually allows us to study another estimator of the variance  $\sigma_0^2$  of the form

$$\hat{\sigma}_{\text{ML},K}^2 = \frac{1}{n} y^\top C_K^{-1} y,$$

where  $(C_K^{-1})_{i,j} = (C^{-1})_{i,j} 1_{|s_i - s_j| \leq K}$ . The proof of Theorem 2 then directly implies the following.

**Corollary 2.** *Let  $K_n$  be any sequence of positive numbers tending to infinity. Let  $\mathcal{L}_{K_n,n}$  be the distribution of  $\sqrt{n}(\hat{\sigma}_{\text{ML},K_n}^2 - \sigma_0^2)$ . Then, as  $n \rightarrow \infty$ ,*

$$d_w(\mathcal{L}_{K_n,n}, \mathcal{N}[0, n \text{Var}(\hat{\sigma}_{\text{ML},K_n}^2 - \sigma_0^2)]) \rightarrow 0.$$

*In addition, we have*

$$n \text{Var}(\hat{\sigma}_{\text{ML},K_n}^2 - \sigma_0^2) - n \text{Var}(\hat{\sigma}_{\text{ML}}^2 - \sigma_0^2) \rightarrow 0.$$

The above corollary shows that one can taper the elements of  $C^{-1}$  when estimating the variance parameter, and obtain the same asymptotic distribution of the error, as long as the taper range goes to infinity, with no rate assumption. This result may have an interest in itself, in view of the existing literature on covariance tapering for Gaussian processes under increasing-domain asymptotics [28, 44]. We also remark that the computation costs of  $\hat{\sigma}_{\text{ML},K}^2$  and  $\hat{\sigma}_{\text{ML}}^2$  have the same orders of magnitude because  $C^{-1}$  needs to be computed in both cases.

**Remark 4.** *In the results of this section, the condition that  $F$  is non-decreasing in Condition 3 ii) can be replaced by the condition that  $C$  has its smallest eigenvalue bounded away from zero. This can be checked in the corresponding proofs.*

## 5. General covariance

### 5.1. Framework

As in Section 3.1, we consider a zero-mean Gaussian process  $Z$  defined on  $\mathbb{R}^d$  with covariance function  $k_Z$  satisfying Condition 2. Let  $Y$  be the random field

defined for any  $s \in \mathbb{R}^d$  by  $Y(s) = T(Z(s))$ , where  $T$  is a fixed function satisfying Condition 3. Furthermore we assume that  $Y$  has zero-mean function and we recall that from Lemma 1, its covariance function  $k_Y$  also satisfies Condition 2. Finally, the sequence of observation locations  $(s_i)_{i \in \mathbb{N}}$  satisfies Condition 1.

Let  $\{k_{Y,\theta}; \theta \in \Theta\}$  be a parametric set of stationary covariance functions on  $\mathbb{R}^d$ , with  $\Theta$  a compact set of  $\mathbb{R}^p$ . We consider the following condition on this parametric set of covariance functions.

**Condition 4.** For all  $s \in \mathbb{R}^d$ ,  $k_{Y,\theta}(s)$  is three times continuously differentiable with respect to  $\theta$ , and we have

$$\sup_{\theta \in \Theta} |k_{Y,\theta}(s)| \leq C_{\sup} \exp(-C_{\inf}|s|), \quad (3)$$

$$\sup_{\substack{\theta \in \Theta \\ \ell=1,2,3 \\ i_1, \dots, i_\ell=1, \dots, p}} \left| \frac{\partial k_{Y,\theta}(s)}{\partial \theta_{i_1}, \dots, \partial \theta_{i_\ell}} \right| \leq \frac{C_{\sup}}{1 + |s|^{d+C_{\inf}}}. \quad (4)$$

The smoothness condition in (4) is classical and is assumed for instance in [7]. As discussed after Condition 2, milder versions of (3) can be assumed for non-transformed Gaussian processes, but (3) is satisfied by most classical families of covariance functions nonetheless.

The next condition, on the Fourier transforms of the covariance functions in the model, is standard.

**Condition 5.** We let  $\hat{k}_{Y,\theta}$  be the Fourier transform of  $k_{Y,\theta}$ . Then  $\hat{k}_{Y,\theta}(f)$  is jointly continuous with respect to  $\theta$  and  $f$  and is strictly positive on  $\Theta \times \mathbb{R}^d$ .

We remark that Condition 5 does not automatically follow from Condition 4 (nor from Conditions 6 to 10 below). For instance, in the case  $d = 1$ , the family of triangular covariance functions  $\{\sigma^2 c_Y; \sigma^2 \in [\sigma_{\inf}^2, \sigma_{\sup}^2]\}$ , with  $0 < \sigma_{\inf}^2 < \sigma_{\sup}^2 < \infty$  and with  $c_Y(x) = (1 - |x|)^+$  for  $x \in \mathbb{R}$ , satisfies Condition 4 but not Condition 5 [12].

Finally, the next condition means that we address the well-specified case [6, 8], where the family of covariance functions does contain the true covariance function of  $Y$ . The well-specified case is considered in the majority of the literature on Gaussian processes.

**Condition 6.** There exists  $\theta_0$  in the interior of  $\Theta$  such that  $k_Y = k_{Y,\theta_0}$ .

In the next two subsections, we study the asymptotic properties of two classical estimators (maximum likelihood and cross validation) for the covariance parameter  $\theta_0$ . The asymptotic properties of these estimators are already known for Gaussian processes and we extend them to the non-Gaussian process  $Y$ .

## 5.2. Maximum Likelihood

For  $n \in \mathbb{N}$ , let  $R_\theta$  be the  $n \times n$  matrix  $(k_{Y,\theta}(s_i - s_j))_{i,j=1,\dots,n}$ , and let

$$\hat{\theta}_{\text{ML}} \in \underset{\theta \in \Theta}{\operatorname{argmin}} L_\theta \quad (5)$$

with

$$L_\theta = \frac{1}{n} (\log(\det(R_\theta)) + y^\top R_\theta^{-1} y)$$

be a maximum likelihood estimator. We will provide its consistency under the following condition.

**Condition 7.** *For all  $\chi > 0$  we have*

$$\liminf_{n \rightarrow \infty} \inf_{\|\theta - \theta_0\| \geq \chi} \frac{1}{n} \sum_{i,j=1}^n (k_{Y,\theta}(s_i - s_j) - k_{Y,\theta_0}(s_i - s_j))^2 > 0.$$

Condition 7 can be interpreted as a global indentifiability condition. It implies in particular that two different covariance parameters yield two different distributions for the observation vector. It is used in several studies, for instance [7].

**Theorem 3.** *Consider the setting of Section 5.1 for  $Z$ ,  $T$ ,  $Y$  and  $(s_i)_{i \in \mathbb{N}}$ . Assume that Conditions 4, 5, 6 and 7 hold. Then, as  $n \rightarrow \infty$ ,*

$$\hat{\theta}_{ML} \xrightarrow{P} \theta_0.$$

**Remark 5.** *Let us discuss and interpret Theorem 3. In Theorem 3, there is a Gaussian process  $Z$  with zero-mean (see Section 6.2 for a discussion of the case where  $Z$  has non-zero mean) and covariance function  $k_Z$ . Consider for the sake of interpretation that the covariance function  $k_Z$  belongs to a set of covariance functions  $\{k_{Z,\alpha}, \alpha \in \mathcal{A}\}$ , with  $k_Z = k_{Z,\alpha_0}$ ,  $\alpha_0 \in \mathcal{A}$ .*

*In Theorem 3 there is a transformation  $T : \mathbb{R} \rightarrow \mathbb{R}$  that is fixed and unknown. The aim of this paper is not to estimate  $T$ , and in particular we do not assume that  $T$  belongs to a known parametric set of transformation functions. This transformation  $T$  then defines the non-Gaussian process  $Y = T(Z)$ . We also assume that  $Y$  has zero-mean, the case of a non-zero mean also being discussed in Section 6.2. In Theorem 3, we assume that the covariance function  $k_Y$  of  $Y$  belongs to the set of covariance functions  $\{k_{Y,\theta}; \theta \in \Theta\}$ , with  $k_Y = k_{Y,\theta_0}$  for  $\theta_0 \in \Theta$ .*

*Assume for the sake of interpretation that, for  $\alpha \in \mathcal{A}$ , if the covariance function of  $Z$  was  $k_{Z,\alpha}$ , then the covariance function of  $Y$  would belong to  $\{k_{Y,\theta}; \theta \in \Theta\}$ . Assume also for the sake of discussion that  $\theta \mapsto k_{Y,\theta}$  is injective (which is an extension of the requirement in Condition 7). Then, there exists a fixed mapping  $\tau : \mathcal{A} \rightarrow \Theta$  such that, if the covariance function of  $Z$  was  $k_{Z,\alpha}$ , then the covariance function of  $Y$  would be  $k_{Y,\tau(\alpha)}$ . The mapping  $\tau$  is uniquely defined because  $\theta \mapsto k_{Y,\theta}$  is assumed to be injective.*

*The process  $Y$  is observed at  $s_1, \dots, s_n$ , yielding the observation vector  $y$  and then the (Gaussian) maximum likelihood estimator  $\hat{\theta}_{ML}$  in Theorem 3. This estimator is based on the false assumption that  $Y$  is a Gaussian process but on the correct assumption that the covariance function of  $Y$  belongs to  $\{k_{Y,\theta}; \theta \in \Theta\}$ . This estimator is thus misspecified from a likelihood point of view but well-specified from a covariance function point of view. Theorem 3 shows that this*

estimator is consistent and Theorem 4 below will show that this estimator is asymptotically unbiased with an asymptotic covariance matrix of sandwich form, see (6). We remark that this form is typical in misspecified models [15, 27, 52].

If the transformation  $T$  was known and injective, one could recover the values  $Z(s_1), \dots, Z(s_n)$  from  $Y(s_1), \dots, Y(s_n)$  and thus compute the (well-specified) Gaussian maximum likelihood estimator  $\hat{\alpha}_{ML}$  of  $\alpha_0$ . Note that we have  $\hat{\theta}_{ML} \neq \tau(\hat{\alpha}_{ML})$  in general. We remark that one can expect  $\hat{\alpha}_{ML}$  to satisfy some type of statistical efficiency principle, as an estimator of  $\alpha_0$ , since it is a well-specified maximum likelihood estimator. Nevertheless, while there is a fair amount of work on efficiency of estimators in the case of i.i.d. data [50], almost no counterparts exist, to the best of our knowledge, for the efficiency of estimators of covariance parameters of Gaussian processes. One of the few examples is [49], in the special case of the exponential covariance functions. If an efficiency property could be proved for  $\hat{\alpha}_{ML}$ , then one could also expect that  $\tau(\hat{\alpha}_{ML})$  satisfies a similar efficiency property, as an estimator of  $\theta_0$  (for instance, in the spirit of Theorem 25.47 in [50]). In particular, one can expect  $\tau(\hat{\alpha}_{ML})$  to be asymptotically a better estimator of  $\theta_0$  than  $\hat{\theta}_{ML}$ . Nevertheless, we emphasize that when  $T$  is unknown,  $\tau(\hat{\alpha}_{ML})$  can not be computed but  $\hat{\theta}_{ML}$  can.

Finally, let us explicit the mapping  $\tau$  in a special case. Consider that the transformation function is given by  $T(x) = x^2 - \mathbb{E}[Z(0)^2]$  for  $x \in \mathbb{R}$ . Consider also that the covariance models  $\{k_{Y,\theta}; \theta \in \Theta\}$  and  $\{k_{Z,\alpha}; \alpha \in \mathcal{A}\}$  are  $\{\sigma^2 e^{-\rho \|\cdot\|}; (\sigma^2, \rho) \in \Theta\}$  and  $\{\sigma^2 e^{-\rho \|\cdot\|}; (\sigma^2, \rho) \in \mathcal{A}\}$ , with  $\Theta, \mathcal{A} \subset (0, \infty)^2$ . Then if  $\alpha_0 = (\sigma_0^2, \rho_0)$ , by Mehler's formula [5], for  $u, v \in \mathbb{R}^d$ ,

$$\text{Cov}(Y(u), Y(v)) = 2 \text{Cov}(Z(u), Z(v))^2 = 2\sigma_0^4 e^{-2\rho_0 \|u-v\|}$$

and thus we have, for  $\alpha = (\alpha_1, \alpha_2)$ ,  $\tau(\alpha) = (2\alpha_1^2, 2\alpha_2)$ .

**Condition 8.** For any  $(\chi_1, \dots, \chi_p) \neq (0, \dots, 0)$ , we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^p \chi_\ell \frac{\partial k_{Y,\theta_0}(s_i - s_j)}{\partial \theta_\ell} \right)^2 > 0.$$

Condition 8 can be interpreted as a regularity condition and as a local identifiability condition around  $\theta_0$ . In the next theorem, we provide the asymptotic normality of the maximum likelihood estimator. In this theorem, the matrices  $M_{\theta_0}$  and  $\Sigma_{\theta_0}$  depend on the number of observation locations.

**Theorem 4.** Consider the setting of Section 5.1 for  $Z$ ,  $T$ ,  $Y$  and  $(s_i)_{i \in \mathbb{N}}$ . Assume that Conditions 4, 5, 6, 7 and 8 hold. Let  $M_{\theta_0}$  be the  $p \times p$  matrix defined by

$$(M_{\theta_0})_{i,j} = \frac{1}{n} \text{tr} \left( R_{\theta_0}^{-1} \frac{\partial R_{\theta_0}}{\partial \theta_i} R_{\theta_0}^{-1} \frac{\partial R_{\theta_0}}{\partial \theta_j} \right).$$

Let  $\Sigma_{\theta_0}$  be the  $p \times p$  covariance matrix defined by  $(\Sigma_{\theta_0})_{i,j} = \text{Cov}(n^{1/2} \partial L_{\theta_0} / \partial \theta_i, n^{1/2} \partial L_{\theta_0} / \partial \theta_j)$ . Let  $\mathcal{L}_{\theta_0,n}$  be the distribution of  $\sqrt{n}(\hat{\theta}_{ML} - \theta_0)$ . Then, as  $n \rightarrow \infty$ ,

$$d_w(\mathcal{L}_{\theta_0,n}, \mathcal{N}[0, M_{\theta_0}^{-1} \Sigma_{\theta_0} M_{\theta_0}^{-1}]) \rightarrow 0. \quad (6)$$

In addition,

$$\limsup_{n \rightarrow \infty} \lambda_1(M_{\theta_0}^{-1} \Sigma_{\theta_0} M_{\theta_0}^{-1}) < +\infty. \quad (7)$$

**Remark 6.** If the sequences of matrices  $M_{\theta_0}$  and  $\Sigma_{\theta_0}$  converge as  $n \rightarrow \infty$ , then  $\sqrt{n}(\hat{\theta}_{ML} - \theta_0)$  converges in distribution to a fixed centered Gaussian distribution where the limiting covariance matrix is given by

$$\lim_{n \rightarrow \infty} M_{\theta_0}^{-1} \Sigma_{\theta_0} M_{\theta_0}^{-1}.$$

Conditions 7 and 8 involve the model of covariance functions  $\{k_{Y,\theta}; \theta \in \Theta\}$  and the sequence of observation locations  $(s_i)_{i \in \mathbb{N}}$  but not the transformation  $T$ . They are further discussed, in a different context, in [13]. We believe that these conditions are mild. For instance, Conditions 7 and 8 hold when the sequence of observation locations  $(s_i)_{i \in \mathbb{N}}$  is a randomly perturbed regular grid, as in [7].

**Lemma 3** (see [7]). For  $i \in \mathbb{N}$ , let  $s_i = g_i + \epsilon_i$ , where  $(g_i)_{i \in \mathbb{N}}$  is a sequence with, for  $N \in \mathbb{N}$ ,  $\{g_1, \dots, g_{Nd}\} = \{(i_1, \dots, i_d); i_1 = 1, \dots, N, \dots, i_d = 1, \dots, N\}$  and where  $(\epsilon_i)_{i \in \mathbb{N}}$  is a sequence of i.i.d. random variables with distribution on  $[-1/2 + \delta, 1/2 - \delta]^d$  with  $0 < \delta < 1/2$ . Then, Condition 7 holds almost surely, provided that, for  $\theta \neq \theta_0$ , there exists  $i \in \mathbb{Z}^d \setminus \{0\}$  for which  $k_{Y,\theta}(i + \epsilon_1 - \epsilon_2)$  and  $k_{Y,\theta_0}(i + \epsilon_1 - \epsilon_2)$  are not almost surely equal.

Furthermore, Condition 8 holds almost surely, provided that for  $(\chi_1, \dots, \chi_p) \neq (0, \dots, 0)$ , there exists  $i \in \mathbb{Z}^d \setminus \{0\}$  for which  $\sum_{\ell=1}^p \chi_\ell \partial k_{Y,\theta_0}(i + \epsilon_1 - \epsilon_2) / \partial \theta_\ell$  is not almost surely equal to zero.

### 5.3. Cross Validation

We consider the cross validation estimator consisting of minimizing the average of the leave-one-out square errors.

Since the leave-one-out errors do not depend on the variance  $k_{Y,\theta}(0)$ , we introduce some additional notation. In Sections 5.3 and 5.4, we let  $\Theta = [\sigma_{\inf}^2, \sigma_{\sup}^2] \times \mathcal{S}$  where  $0 < \sigma_{\inf}^2 < \sigma_{\sup}^2 < \infty$  are fixed and where  $\mathcal{S}$  is compact in  $\mathbb{R}^{p-1}$ . We let  $\theta = (\sigma^2, \psi)$  with  $\sigma_{\inf}^2 \leq \sigma^2 \leq \sigma_{\sup}^2$  and  $\psi \in \mathcal{S}$ . We assume that for  $\theta \in \Theta$ ,  $k_{Y,\theta} = \sigma^2 c_{Y,\psi}$ , with  $c_{Y,\psi}$  a stationary correlation function. Similarly, we let  $\theta_0 = (\sigma_0^2, \psi_0)$ . For  $\psi \in \mathcal{S}$ , we let  $C_\psi = (c_{Y,\psi}(s_i - s_j))_{i,j=1,\dots,n}$ . Cross validation is defined for  $n \in \mathbb{N}$  by

$$\hat{\psi}_{CV} \in \operatorname{argmin}_{\psi \in \mathcal{S}} CV_\psi, \quad (8)$$

with

$$CV_\psi = \frac{1}{n} y^\top C_\psi^{-1} \operatorname{diag}(C_\psi^{-1})^{-2} C_\psi^{-1} y,$$

where  $\operatorname{diag}(M)$  is obtained by setting the off-diagonal elements of a square matrix  $M$  to zero. The criterion  $CV_\psi$  is the average of the leave-one-out square errors, as is shown for instance in [25, 6]. More precisely, if we let  $\hat{y}_{i,\psi}$  be the

best linear predictor of  $y_i$  based on  $(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ , assuming that  $Y$  has covariance function  $\sigma^2 c_{Y,\psi}$ , for any choice of  $\sigma^2 \in [\sigma_{\inf}^2, \sigma_{\sup}^2]$ , then we have

$$CV_\psi = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_{i,\psi})^2.$$

In particular, we point out that  $\hat{y}_{i,\psi}$  does not depend on  $\sigma^2$ , because the best linear predictors from the two covariance functions  $\sigma_1^2 c_{Y,\psi}$  and  $\sigma_2^2 c_{Y,\psi}$  are the same, for any  $\sigma_1^2, \sigma_2^2 \in [\sigma_{\inf}^2, \sigma_{\sup}^2]$ .

Let us insist on the fact that we consider here a cross validation estimator of the correlation parameter  $\psi_0$  but that we do not consider a cross validation estimator of  $\sigma_0^2$ . Indeed, we consider a cross validation estimator obtained by minimizing the average of the leave-one-out square errors, that depends only on  $\psi$ , not on  $\sigma^2$ .

The asymptotic behaviour of  $\hat{\psi}_{CV}$  was studied in the Gaussian framework in [7] and under increasing-domain asymptotics.

The next identifiability condition is also made in [7].

**Condition 9.** For all  $\chi > 0$ , we have

$$\liminf_{n \rightarrow \infty} \inf_{\|\psi - \psi_0\| \geq \chi} \frac{1}{n} \sum_{i,j=1}^n (c_{Y,\psi}(s_i - s_j) - c_{Y,\psi_0}(s_i - s_j))^2 > 0.$$

The next theorem provides the consistency of the cross validation estimator.

**Theorem 5.** Consider the setting of Section 5.1 for  $Z$ ,  $T$ ,  $Y$  and  $(s_i)_{i \in \mathbb{N}}$ . Assume that Conditions 4, 5, 6 and 9 hold. Then, as  $n \rightarrow \infty$ ,

$$\hat{\psi}_{CV} \xrightarrow{P} \psi_0.$$

The next condition is a local identifiability condition.

**Condition 10.** For any  $(\chi_1, \dots, \chi_{p-1}) \neq (0, \dots, 0)$ , we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^{p-1} \chi_\ell \frac{\partial}{\partial \psi_\ell} (c_{Y,\psi_0}(s_i - s_j)) \right)^2 > 0.$$

In the next theorem, we provide the asymptotic normality of the cross validation estimator. In this theorem, the matrices  $N_{\psi_0}$  and  $\Gamma_{\psi_0}$  depend on the number of observation locations.

**Theorem 6.** Consider the setting of Section 5.1 for  $Z$ ,  $T$ ,  $Y$  and  $(s_i)_{i \in \mathbb{N}}$ . Assume that Conditions 4, 5, 6, 9 and 10 hold. Let  $N_{\psi_0}$  be the  $(p-1) \times (p-1)$  matrix defined by

$$(N_{\psi_0})_{i,j} = -\frac{8}{n} \operatorname{tr} \left( \frac{\partial C_{\psi_0}}{\partial \psi_j} C_{\psi_0}^{-1} \operatorname{diag}(C_{\psi_0}^{-1})^{-3} \operatorname{diag} \left( C_{\psi_0}^{-1} \frac{\partial C_{\psi_0}}{\partial \psi_i} C_{\psi_0}^{-1} \right) C_{\psi_0}^{-1} \right)$$

$$\begin{aligned}
& + \frac{2}{n} \operatorname{tr} \left( \frac{\partial C_{\psi_0}}{\partial \psi_j} C_{\psi_0}^{-1} \operatorname{diag}(C_{\psi_0}^{-1})^{-2} C_{\psi_0}^{-1} \frac{\partial C_{\psi_0}}{\partial \psi_i} C_{\psi_0}^{-1} \right) \\
& + \frac{6}{n} \operatorname{tr} \left( \operatorname{diag}(C_{\psi_0}^{-1})^{-4} \operatorname{diag} \left( C_{\psi_0}^{-1} \frac{\partial C_{\psi_0}}{\partial \psi_i} C_{\psi_0}^{-1} \right) \operatorname{diag} \left( C_{\psi_0}^{-1} \frac{\partial C_{\psi_0}}{\partial \psi_j} C_{\psi_0}^{-1} \right) C_{\psi_0}^{-1} \right).
\end{aligned}$$

Let  $\Gamma_{\psi_0}$  be the  $(p-1) \times (p-1)$  covariance matrix defined by

$$(\Gamma_{\psi_0})_{i,j} = \operatorname{Cov}(n^{1/2} \partial C V_{\psi_0} / \partial \psi_i, n^{1/2} \partial C V_{\psi_0} / \partial \psi_j).$$

Let  $\mathcal{Q}_{\psi_0,n}$  be the distribution of  $\sqrt{n}(\hat{\psi}_{CV} - \psi_0)$ . Then, as  $n \rightarrow \infty$ ,

$$d_w \left( \mathcal{Q}_{\psi_0,n}, \mathcal{N}[0, N_{\psi_0}^{-1} \Gamma_{\psi_0} N_{\psi_0}^{-1}] \right) \rightarrow 0. \quad (9)$$

In addition,

$$\limsup_{n \rightarrow \infty} \lambda_1(N_{\psi_0}^{-1} \Gamma_{\psi_0} N_{\psi_0}^{-1}) < +\infty. \quad (10)$$

Similarly as for maximum likelihood, Condition 9 is a global identifiability condition for the correlation function. In the same way, Condition 10 is a local identifiability condition for the correlation function around  $\psi_0$ .

We remark that the conditions for cross validation imply those for maximum likelihood.

**Lemma 4.** *Condition 9 implies Condition 7 and Condition 10 implies Condition 8.*

Finally, similarly as for maximum likelihood, we point out that Conditions 9 and 10 hold in the case of a randomly perturbed regular grid, as in [7].

**Lemma 5** (see [7]). *Let  $(s_i)_{i \in \mathbb{N}}$  and  $(\epsilon_i)_{i \in \mathbb{N}}$  be as in Lemma 3. Then, Condition 9 holds almost surely, provided that, for  $\psi \neq \psi_0$ , there exists  $i \in \mathbb{Z}^d \setminus \{0\}$  for which  $c_{Y,\psi}(i + \epsilon_1 - \epsilon_2)$  and  $c_{Y,\psi_0}(i + \epsilon_1 - \epsilon_2)$  are not almost surely equal.*

*Furthermore, Condition 10 holds almost surely, provided that for  $(\chi_1, \dots, \chi_{p-1}) \neq (0, \dots, 0)$ , there exists  $i \in \mathbb{Z}^d \setminus \{0\}$  for which  $\sum_{\ell=1}^{p-1} \chi_\ell \partial c_{Y,\psi_0}(i + \epsilon_1 - \epsilon_2) / \partial \psi_\ell$  is not almost surely equal to zero.*

**Remark 7.** *Here we have provided the asymptotic properties of cross validation by minimization of the average of the square leave-one-out errors for estimating  $\psi_0$ . This corresponds to an extension of [7] to the transformed Gaussian framework. We remark that there exist other cross validation methods that provide estimators of both  $\psi_0$  and  $\sigma_0^2$  [6, 14, 40, 59] but their asymptotic properties are not established under increasing-domain asymptotics, even for Gaussian processes (note that [14] provides the fixed-domain asymptotic properties of one of these other cross validation methods for the exponential covariance function in dimension one). It would be interesting to extend the results of Section 5.3 to these other cross validation methods in future work.*



#### 5.4. Joint asymptotic normality

From Theorems 4 and 6, both the maximum likelihood and cross validation estimators converge at the standard parametric rate  $n^{1/2}$ . Let us write  $\hat{\theta}_{\text{ML}} = (\hat{\sigma}_{\text{ML}}^2, \hat{\psi}_{\text{ML}})$ . In the case where  $T$  is the identity function (that is, where we observe Gaussian processes instead of transformed Gaussian processes), numerical experiments tend to show that  $\hat{\psi}_{\text{ML}}$  is more accurate than  $\hat{\psi}_{\text{CV}}$  [7]. Indeed, when  $T$  is the identity function, maximum likelihood is based on the Gaussian probability density function of the observation vector.

In contrast, when  $T$  is not the identity function,  $\hat{\psi}_{\text{ML}}$  is an M-estimator based on a criterion which does not coincide with the observation probability density function anymore. Hence, it is conceivable that  $\hat{\psi}_{\text{CV}}$  could become more accurate than  $\hat{\psi}_{\text{ML}}$ . Furthermore, it is possible that using linear combinations of these two estimators could result in a third one with improved accuracy [35, 16].

Motivated by this discussion, we now provide a joint central limit theorem for the maximum likelihood and cross validation estimators.

**Theorem 7.** *Consider the setting of Section 5.1 for  $Z$ ,  $T$ ,  $Y$  and  $(s_i)_{i \in \mathbb{N}}$ . Assume that Conditions 4, 5, 6, 9 and 10 hold. Let  $D_{\theta_0}$  be the  $(2p-1) \times (2p-1)$  block diagonal matrix with first  $p \times p$  block equal to  $M_{\theta_0}$  and second  $(p-1) \times (p-1)$  block equal to  $N_{\psi_0}$ , with the notation of Theorems 4 and 6. Also let  $\Psi_{\theta_0}$  be the  $(2p-1) \times (2p-1)$  covariance matrix of the vector  $n^{1/2}(\partial L_{\theta_0}/\partial \theta, \partial CV_{\psi_0}/\partial \psi)$ .*

*Let  $\mathcal{Q}_{\theta_0, n}$  be the distribution of*

$$\sqrt{n} \begin{pmatrix} \hat{\theta}_{\text{ML}} - \theta_0 \\ \hat{\psi}_{\text{CV}} - \psi_0 \end{pmatrix}.$$

*Then, as  $n \rightarrow \infty$ ,*

$$d_w(\mathcal{Q}_{\theta_0, n}, \mathcal{N}[0, D_{\theta_0}^{-1} \Psi_{\theta_0} D_{\theta_0}^{-1}]) \rightarrow 0. \quad (11)$$

*In addition,*

$$\limsup_{n \rightarrow \infty} \lambda_1(D_{\theta_0}^{-1} \Psi_{\theta_0} D_{\theta_0}^{-1}) < +\infty. \quad (12)$$

**Remark 8.** *From Theorem 7, considering any  $C^1$  function  $f$  from  $\mathcal{S}^2 \rightarrow \mathcal{S}$  such that  $f(\psi, \psi) = \psi$  for any  $\psi \in \mathcal{S}$ , and applying the classical delta method, we obtain the asymptotic normality of the new estimator  $f(\hat{\psi}_{\text{ML}}, \hat{\psi}_{\text{CV}})$ . A classical choice for  $f$  is  $f(\psi_1, \psi_2) = \lambda\psi_1 + (1-\lambda)\psi_2$ , which leads to linear aggregation [35, 16]. We remark that selecting an optimal  $\lambda$  leading to the smallest asymptotic covariance matrix necessitates an estimation of the asymptotic covariance matrix in Theorem 7. We leave this as an open problem for further research.*

**Remark 9.** *In all the results of Section 5, the condition that  $F$  is non-decreasing in Condition 3 ii) is actually not needed. It can indeed be verified that all the proofs for Section 5 do not use this condition. In fact, Condition 3 ii) is used in Lemma 1 to show that covariance matrices have their smallest eigenvalues bounded away from zero. This is already the case in Section 5, thanks to Condition 5.*

**Remark 10.** *The results and proofs in Section 5 rely significantly on the assumption that the set of covariance parameters  $\Theta$  is compact. This is a common assumption in the literature [7, 34, 44]. Nevertheless, this assumption is, in some aspects, unnatural. For instance, the domain of allowed values of estimators of the (constant) variance of a stationary random field is often  $[0, \infty)$ .*

*An extension of the results of this paper to non-compact sets  $\Theta$  of covariance parameters, while valuable, could turn out to be very challenging, and would definitely require new proof arguments. For instance, we remark that letting correlation length parameters go to zero or infinity may yield asymptotic situations and techniques that are qualitatively different from the ones tackled in this paper, see for instance [42, 54].*

## 6. Discussions of extensions

### 6.1. Nugget effect

In Section 5, Condition 5 implies that the covariance function  $k_{Y,\theta}$  is continuous for any  $\theta \in \Theta$ . Furthermore the true covariance function  $k_Y = k_{Y,\theta_0}$  of  $Y$  is also continuous. Hence, the results of Section 5 hold for models of continuous covariance functions and with error-free observations of continuous transformed Gaussian processes.

Here, we provide an extension of the results of Section 5 to a nugget effect on the covariance function  $k_{Y,\theta}$ . We assume that for  $\theta \in \Theta$ , the covariance function  $k_{Y,\theta}$  is of the form  $k_{Y,\theta}(s) = \bar{k}_{Y,\theta}(s) + \eta_\theta 1_{s=0}$ ,  $s \in \mathbb{R}^d$ , where  $\bar{k}_{Y,\theta}$  is a continuous stationary covariance function on  $\mathbb{R}^d$  and  $\eta_\theta \geq 0$ . In this case, Condition 5 does not hold, but it can be replaced by the following condition.

**Condition 11.** *We have  $\inf_{\theta \in \Theta} \eta_\theta > 0$ .*

We also assume that, for  $s \in \mathbb{R}^d$ ,  $Y(s) = T(Z(s)) + \check{N}(s)$ , where  $Y$  and  $Z$  still have zero-mean and where there exists  $\check{T} : \mathbb{R} \rightarrow \mathbb{R}$  satisfying Condition 3, such that, for  $s \in \mathbb{R}^d$ ,  $\check{N}(s) = \check{T}(\zeta(s))$ , where  $\zeta$  is a centered Gaussian random field, independent of  $Z$ , with  $\text{Cov}(\zeta(u), \zeta(v)) = \kappa 1_{u=v}$  for  $u, v \in \mathbb{R}^d$ , with  $\kappa > 0$ . We also assume that  $\check{N}$  has zero-mean. We still assume that  $k_Z$  satisfies Condition 2. Note that  $k_Z$  is thus not necessarily continuous and can also include a nugget effect as described in Remark 2.

In this setting, the results of Section 5 can be extended in a relatively straightforward way.

**Proposition 1.** *Consider the setting of Section 6.1. Let Condition 5 be replaced by Condition 11. Let Conditions 4, 6, 7, 8, 9 and 10 be unchanged and applied to the new form of  $k_{Y,\theta}$  given in this section. Then Theorems 3, 4, 5, 6 and 7 still hold.*

We remark that in Proposition 1 we still address the well-specified case. In particular, there exists  $\theta_0 \in \Theta$  such that

$$k_{Y,\theta_0}(0) = \bar{k}_{Y,\theta_0}(0) + \eta_{\theta_0} = \text{Var}(T(Z(s))) + \text{Var}(\check{T}(\zeta(s))),$$

for all  $s \in \mathbb{R}^d$ . We also remark that the setting considered here is equivalent to additive i.i.d. observation errors on a transformed Gaussian process, since the observation points are two-by-two distinct.

## 6.2. Unknown mean

Throughout Section 5, we assume that the mean function of  $Y$  is constant, known and equal to zero (in particular, the maximum likelihood estimator in (5) is based on the density of a centered Gaussian vector). The results of Section 5 also apply to the case of any constant known mean  $\nu_0 \in \mathbb{R}$  for  $Y$ , since one can then also observe the centered process  $Y - \nu_0$ . However, the results of Section 5 do not apply to the case of a constant unknown mean  $\nu_0$  for  $Y$ .

In fact, even for (non-transformed) Gaussian processes, there exist very few results on the joint estimation of a constant mean and of covariance parameters by maximum likelihood or cross validation. For instance, the reference [7], that we extend from the Gaussian to transformed Gaussian case, considers a constant known mean. Among these few results, one can mention [22, 36], but the conditions there are more difficult to check and to interpret than those given in [7].

In future work, it would be interesting to extend the results of Section 5 to transformed Gaussian processes with constant unknown mean. For instance, one may consider a likelihood criterion to minimize of the form

$$\frac{1}{n} (\log(\det(R_\theta)) + (y - v_\nu)^\top R_\theta^{-1} (y - v_\nu)),$$

where  $\theta$  is still the covariance parameter,  $\nu$  is the mean parameter and  $v_\nu = (\nu, \dots, \nu)^\top$  is of size  $n$ . In this case a first task is to extend the results of [7] to show the consistency and asymptotic normality of the maximum likelihood estimators  $\hat{\theta}_{\text{ML}}$  and  $\hat{\nu}_{\text{ML}}$  when  $Y$  is Gaussian. A second task is to further extend these results to the case where  $Y$  is transformed Gaussian.

Finally, in the case where  $Z$  or  $Y$  has non-zero constant mean, Theorem 2 can be extended as follows.

**Corollary 3.** *Consider the setting of Theorem 2, with the only modification that the Gaussian process  $Z$  has a constant mean function  $\mu_0 \in \mathbb{R}$  and that the transformed process  $Y = T(Z)$  has a constant mean function  $\nu_0 \in \mathbb{R}$ . Then the conclusion of Theorem 2 holds, with  $y$  replaced by  $y - v_{\nu_0}$ .*

This extension of Theorem 2 can be relevant and useful to the question of extending the results of Section 5, as discussed above.

## 7. Illustration

In this section we numerically illustrate the convergence of different estimators as stated in Corollary 1 and Theorems 4, 6 and 7. We use the following simulation setup in dimension  $d = 2$ . For  $n = 4(m + 1/2)^2$  we define the grid

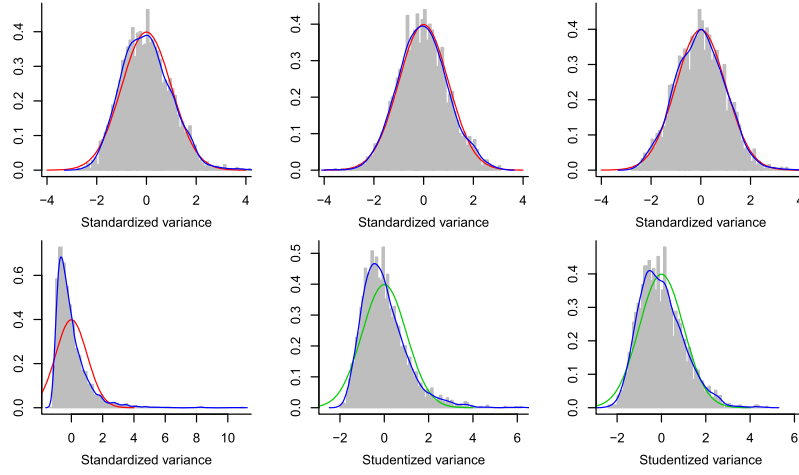


FIG 1. Histograms of standardized and studentized variance maximum likelihood estimates. Blue: empirical density (kernel density estimates), red: asymptotic density, green: asymptotic density based on the empirical variance. The top row shows results for the process  $Z$  (the Gaussian case), the bottom row for the transformed process  $Y = Z^2 - \sigma_0^2$  (the non-Gaussian case). The columns are for  $n = 100, 400, 900$ . All panels are based on  $N = 2500$  replicates.

$\{-m, \dots, m\}^2$  and add i.i.d. variables with uniform distribution on  $[-0.4, 0.4]^2$  to obtain  $n$  observation points. Thus, we have a distance of at least  $\Delta = 0.2$  between the individual observation locations. The zero-mean Gaussian process  $Z$  has stationary covariance function  $k_Z(s) = \sigma_0^2 \exp(-\|s\|/\rho_0)$ ,  $s \in \mathbb{R}^2$  and we will denote this reference case as the Gaussian case throughout. We define the zero-mean process  $Y = T(Z) = Z^2 - \sigma_0^2$  and we will denote this case as the non-Gaussian case. Recall that  $k_Y(s) = 2k_Z(2s) = 2\sigma_0^4 \exp(-2\|s\|/\rho_0)$  (see Remark 5). We set the marginal variance to  $\sigma_0^2 = 1.5$  and the range to  $\rho_0 = 2$ . Hence, in the non-Gaussian case, the marginal variance of  $Y$  is  $2\sigma_0^4 = 4.5$  with a range of  $\rho_0/2$  which is equal to a half of that of  $Z$ .

To start, we consider the maximum likelihood estimates of the marginal variance parameters, when the range of  $Y$  or  $Z$  is assumed to be known, i.e., Corollary 1. Figure 1 illustrates the empirical densities of  $\mathcal{L}_n$  in Corollary 1 for  $n = 100, 400$  and  $900$  observation locations based on  $N = 2500$  replicates. For moderate  $n$  sizes and in the non-Gaussian case, the asymptotic variance  $\sigma_\infty^2 := n \text{Var}(\hat{\sigma}_{\text{ML}}^2 - 2\sigma_0^4)$  (Corollary 1) can be calculated based on (26) and using

$$\begin{aligned} \text{Cov}(y_i y_j, y_k y_l) &= 4(k_{i,k}^2 k_{j,l}^2 + k_{i,l}^2 k_{j,k}^2) \\ &\quad + 16(k_{i,k} k_{i,l} k_{j,k} k_{j,l} + k_{i,j} k_{i,l} k_{j,k} k_{k,l} + k_{i,j} k_{i,k} k_{j,l} k_{k,l}), \end{aligned}$$

with  $k_{i,j} = k_Z(s_i - s_j)$ . The above display follows from tedious computations based on Isserlis' theorem. The densities in red in Figure 1 are based on the

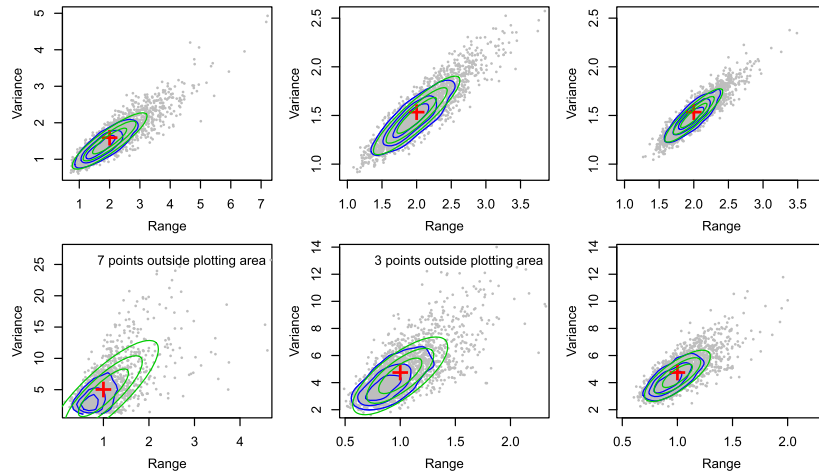


FIG 2. Scatter plots of variance and range maximum likelihood estimates. Blue: contour lines of kernel density estimates; green: contour lines of asymptotic density based on the empirical bivariate covariance matrix; red cross: true mean (true parameter values). The top row shows results for the process  $Z$  (Gaussian case), the bottom row for the transformed process  $Y = Z^2 - \sigma_0^2$  (non-Gaussian case). The columns are for  $n = 100, 400, 900$ . All panels are based on  $N = 2500$  replicates.

asymptotic distribution  $\mathcal{N}[0, \sigma_\infty^2]$ . In the non-Gaussian case and for  $n \geq 400$ , the calculation of  $\sigma_\infty^2$  is computationally prohibitive, so  $\sigma_\infty^2$  has instead been approximated by the empirical variance with the corresponding densities indicated in green. As expected, the convergence for the Gaussian case is faster than for the non Gaussian case. But in both situations, the results behave nicely.

We now turn to Theorem 4 and consider the bivariate variance and range maximum likelihood estimation. That is, we consider the two-dimensional maximum likelihood estimates of  $(\sigma_0^2, \rho_0)$  in the Gaussian case and of  $(2\sigma_0^4, \rho_0/2)$  in the non-Gaussian case. Again, we do not observe many surprises. Skewness of the empirical distribution is slightly higher compared to the single variance parameter estimation, and convergence is slightly slower, as is illustrated in Figure 2.

For the general setting, when estimating jointly the variance and range parameter, the asymptotic bivariate covariance matrix is challenging to compute (see Theorem 4) and thus Figure 2 illustrates the empirical densities and densities based on the empirical bivariate covariance matrix.

We now consider not only maximum likelihood estimation of the variance and range, but also cross validation estimation of the range (see the beginning of Section 5.3). We have observed that the range estimates based on cross validation are much more variable, and in many situations the maximum was attained at the (imposed) boundary. Here we used the bound  $[2/15, 12]$ , i.e., smaller than the minimal distance between two observation locations and 6 (resp. 12) times

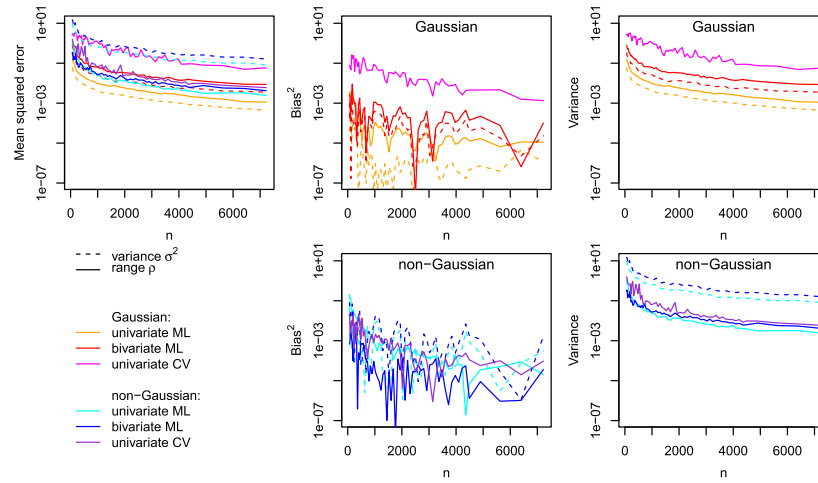


FIG 3. Mean squared error (top left), squared bias (middle column) and variance (right column) as a function of  $n$  for different settings in log-scale. The variance parameter is represented in dashed lines, the range parameter in solid lines. Process  $Z$  is shown by reddish colors (Gaussian case), transformed process  $Y = Z^2 - \sigma_0^2$  in blueish colors (non-Gaussian case). The panels are based on  $N = 250$  replicates that did not reveal any convergence issues.

the diameter of the observation points of  $Z$  (resp.  $Y$ ). Estimates at or close to the boundary indicate convergence issues and would imply a second, possibly manual, inspection. For the reported results, we eliminated all cross validation cases that yielded estimates outside  $[0.14, 11.4]$ .

Figure 3 shows the mean squared error, squared bias and variance of  $\hat{\sigma}_{\text{ML}}^2$ ,  $\hat{\rho}_{\text{ML}}$  and  $\hat{\rho}_{\text{CV}}$  under different settings. For maximum likelihood, we consider the univariate case (one parameter is estimated while the other is known) and the bivariate case (both parameters are jointly estimated). For cross validation, only the range parameter is estimated (see the beginning of Section 5.3), and thus only the univariate case is considered. The mean squared error is dominated by the variance component. Univariate maximum likelihood estimation for Gaussian cases have low bias and the lowest variance (top left and right panel). Joint maximum likelihood estimation has a somewhat larger variance than individual estimation. Surprisingly, cross validation for Gaussian cases has a higher variance compared to cross validation for non-Gaussian cases.

Recall that Theorems 4, 6 and 7 show that, as  $n$  increases, the distribution of the standardized estimation error is close to a Gaussian distribution in terms of the metric  $d_w$ . In Figure 4, we illustrate this by computing one-dimensional Wasserstein distances between the empirical distribution of the standardized estimation errors and Gaussian distributions. The figure shows the Wasserstein distance ( $p = 1$ ) as a function of the number of observation locations  $n$  for individual parameters and for specific bivariate settings (similarly as for Figure 3). In each case, the samples have been centered around the true mean

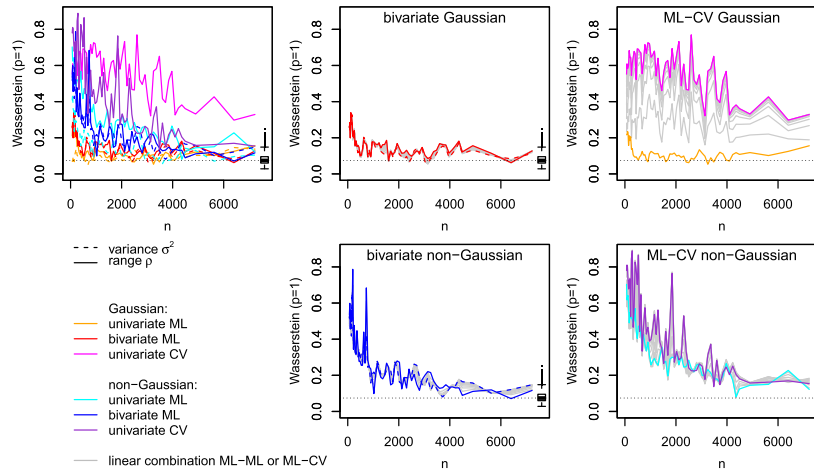


FIG 4. Wasserstein distance ( $p = 1$ ). Top left: marginal for each parameter. Center panels: bivariate estimation of range and variance by maximum likelihood as in Theorem 4 and linear combinations thereof; right panels: univariate estimation of the range parameter by maximum likelihood and cross validation and with linear combinations of these two estimators as in Theorem 7. Top middle and right panels: Gaussian cases. Lower row panels: non-Gaussian case. The colors and line styles follow those in Figure 3. The gray lines are Wasserstein distances for estimates based on linear combinations of maximum likelihood (center column) and of maximum likelihood and cross validation (right column). The panels are based on  $N = 250$  replicates that did not reveal any convergence issues. The small boxplot on the right in each panel shows the Wasserstein distance for sample size  $n = 250$  of 10000 realizations of  $N[0, 1]$ ; the horizontal dotted line shows the median thereof.

(true parameter values) and standardized by an empirical standard deviation ( $n$ -weighted average over all the samples). Their empirical distribution is compared to the standardized Gaussian distribution. The top left panel shows that the densities of the cross validation parameters are converging slowest whereas their mean squared error is comparable (see Figure 3); the densities are highly skewed and thus lead to much larger Wasserstein distances compared to the distributions of the maximum likelihood derived parameters. For the bivariate maximum likelihood estimation the marginal distributions have very similar Wasserstein distances; in the center panels: the dashed and solid colored lines are visually hardly separable. As suggested by the individual panels of Figures 1 and 2, convergence in the Gaussian case is much faster compared to the non-Gaussian case. The right column of Figure 4 illustrates the joint asymptotic normality of the range parameter estimators by maximum likelihood and cross validation. The gray lines there illustrate Theorem 7 and are Wasserstein distances for linear combinations of the range estimates by maximum likelihood and cross validation, i.e.,  $\lambda \hat{p}_{\text{ML}} + (1 - \lambda) \hat{p}_{\text{CV}}$  for  $\lambda = j/10$ ,  $j = 1, \dots, 9$ . The highly skewed distribution of the cross validation-estimated range parameter for Gaussian processes is clearly visible. In the non Gaussian case, the effect of the skewness is less pronounced since the maximum likelihood is skewed as well.

## 8. Conclusion

We have shown that the covariance parameters of transformed Gaussian processes can be estimated by cross validation and Gaussian maximum likelihood, with the same rate of convergence as in the case of non-transformed Gaussian processes. In particular, Gaussian maximum likelihood works well asymptotically, despite the fact that the observations do not have a Gaussian distribution. Hence Gaussian maximum likelihood is here robust with respect to non-Gaussian data. This provides the first step of a theoretical validation of the use of Gaussian maximum likelihood in frequent cases where the data are non-Gaussian.

In future research, it would be interesting to extend the results of this paper to other classes of non-Gaussian random fields rather than only transformed Gaussian processes. In addition, the asymptotic analysis of estimators of the transformation of transformed Gaussian processes is of great interest.

## Appendix A: Proofs

### A.1. Technical results

**Lemma 6.** *Let  $q \in \mathbb{N}$  be fixed. Let  $g : \mathbb{R}^q \rightarrow \mathbb{R}^+$  be fixed and satisfy  $g(x) \leq C_{\text{sup}} \exp(C_{\text{sup}}|x|)$ . Let  $W$  be a Gaussian vector of dimension  $q$ . Then  $\mathbb{E}[g(W)] < \infty$ .*

*Proof.* Without loss of generality, we can assume that  $W_i$  has variance 1 for  $i = 1, \dots, q$ . We let  $w_i$  be the mean of  $W_i$  for  $i = 1, \dots, q$ . We have, for  $t > 1$ ,

$$\begin{aligned} \mathbb{P}(g(W) \geq t) &\leq \mathbb{P}(C_{\text{sup}} \exp(C_{\text{sup}} \max_{i=1, \dots, q} |W_i|) \geq t) \\ &\leq \sum_{i=1}^q \mathbb{P}(C_{\text{sup}} \exp(C_{\text{sup}} |W_i|) \geq t) \\ &= \sum_{i=1}^q \mathbb{P}(|W_i| \geq (1/C_{\text{sup}}) \log(t/C_{\text{sup}})) \\ &\leq 2 \sum_{i=1}^q \mathbb{P}(W \geq (1/C_{\text{sup}}) \log(t/C_{\text{sup}}) - |w_i|), \end{aligned}$$

where  $W \sim \mathcal{N}[0, 1]$ . From the Gaussian tail inequality, we obtain, for

$$t \geq C_{\text{sup}} \exp(C_{\text{sup}} (\max_{i=1, \dots, q} |w_i| + 1)),$$

that

$$\mathbb{P}(g(W) \geq t) \leq \frac{2q}{\sqrt{2\pi}} \exp\left(-(1/2)\left((1/C_{\text{sup}}) \log(t/C_{\text{sup}}) - \max_{i=1, \dots, q} |w_i|\right)^2\right).$$

The function of  $t$  above is clearly summable as  $t \rightarrow +\infty$ . Hence, we have  $\mathbb{E}[g(W)] = \int_0^\infty \mathbb{P}(g(W) \geq t) < +\infty$ .  $\square$



**Lemma 7.** Let  $X$  be a centered Gaussian process with covariance function  $k_X$  satisfying Condition 2. Let  $F$  satisfy Condition 3. Let  $W$  be the spatial process  $F(X(\cdot))$  and assume that  $W$  is centered. Let  $(x_i)_{i \in \mathbb{N}}$  satisfy Condition 1.

Then, we have, for any  $r_1, r_2 \in \mathbb{N}$  and  $\Delta \geq 0$ ,

$$\sup_{\substack{i_1, \dots, i_{r_1} \in \mathbb{N} \\ j_1, \dots, j_{r_2} \in \mathbb{N} \\ \min_{a=1, \dots, r_1, b=1, \dots, r_2} |x_{i_a} - x_{j_b}| \geq \Delta}} |\text{Cov}(W(x_{i_1}) \dots W(x_{i_{r_1}}), W(x_{j_1}) \dots W(x_{j_{r_2}}))| \leq C_{\text{sup}} e^{-C_{\text{inf}} \Delta},$$

where  $C_{\text{sup}}$  and  $C_{\text{inf}}$  depend on  $r_1, r_2$  but not on  $\Delta$ .

*Proof.* Let  $\Delta \geq 0$ ,  $i_1, \dots, i_{r_1} \in \mathbb{N}$  and  $j_1, \dots, j_{r_2} \in \mathbb{N}$  such that

$$\min_{a=1, \dots, r_1, b=1, \dots, r_2} |x_{i_a} - x_{j_b}| \geq \Delta.$$

Let  $\epsilon_3 \sim \mathcal{N}[0, I_{r_1}]$  and  $\epsilon_4 \sim \mathcal{N}[0, I_{r_2}]$ ,  $\epsilon_3$  and  $\epsilon_4$  being independent. Let  $R$  be the  $r_2 \times r_1$  matrix  $(k_X(x_{j_a}, x_{i_b}))_{a,b}$ , let  $C_1$  be the  $r_1 \times r_1$  matrix  $(k_X(x_{i_a}, x_{i_b}))_{a,b}$  and let  $C_2$  be the  $r_2 \times r_2$  matrix  $(k_X(x_{j_a}, x_{j_b}))_{a,b}$ . Let  $M = RC_1^{-1}$ . Let  $K$  be a matrix square root of  $C_2 - RC_1^{-1}R^\top$ . Let  $K_1$  be the unique symmetric matrix square root of  $C_1$ .

Then the vector  $((K_1\epsilon_3)^\top, (MK_1\epsilon_3 + K\epsilon_4)^\top)$  has the same distribution as

$$(X(x_{i_1}), \dots, X(x_{i_{r_1}}), X(x_{j_1}), \dots, X(x_{j_{r_2}})).$$

For  $i = 1, 2$ , for  $x = (x_1, \dots, x_{r_i}) \in \mathbb{R}^{r_i}$ , let  $f_i(x) = F(x_1) \dots F(x_{r_i}) \in \mathbb{R}$ . Then we have

$$\begin{aligned} & \text{Cov}(W(x_{i_1}) \dots W(x_{i_{r_1}}), W(x_{j_1}) \dots W(x_{j_{r_2}})) \\ &= \text{Cov}(f_1(K_1\epsilon_3), f_2(MK_1\epsilon_3 + K\epsilon_4)). \end{aligned}$$

By a Taylor expansion, there exists a random vector  $\epsilon_5$  belonging to the segment with endpoints  $K\epsilon_4$  and  $MK_1\epsilon_3 + K\epsilon_4$  such that, with  $G_2(\epsilon_5)$  the gradient column vector of  $f_2$  at  $\epsilon_5$ , we have

$$f_2(MK_1\epsilon_3 + K\epsilon_4) = f_2(K\epsilon_4) + (MK_1\epsilon_3)^\top G_2(\epsilon_5).$$

This yields

$$\begin{aligned} & |\text{Cov}(W(x_{i_1}) \dots W(x_{i_{r_1}}), W(x_{j_1}) \dots W(x_{j_{r_2}}))| \\ &= |\text{Cov}(f_1(K_1\epsilon_3), \epsilon_3^\top K_1^\top M^\top G_2(\epsilon_5))| \\ &\leq \sqrt{\mathbb{E}[f_1^2(K_1\epsilon_3)]} \sqrt{\mathbb{E}[(\epsilon_3^\top K_1^\top M^\top G_2(\epsilon_5))^2]}. \end{aligned}$$

From Condition 2 ii) and from the equivalence of norms, we obtain  $\|M\|_{op} \leq C_{\text{sup}} e^{-C_{\text{inf}} \Delta}$  and  $\|K_1\|_{op} \leq C_{\text{sup}}$ , where  $C_{\text{sup}}$  and  $C_{\text{inf}}$  do not depend on

$$i_1, \dots, i_{r_1}, j_1, \dots, j_{r_2}, \Delta.$$

By equivalence of norms, we then obtain, with  $C_{\sup}$  and  $C_{\inf}$  not depending on  $i_1, \dots, i_{r_1}, j_1, \dots, j_{r_2}, \Delta$ ,

$$\begin{aligned} & |\text{Cov}(W(x_{i_1}) \dots W(x_{i_{r_1}}), W(x_{j_1}) \dots W(x_{j_{r_2}}))| \\ & \leq C_{\sup} e^{-C_{\inf} \Delta} \sqrt{\mathbb{E}[f_1^2(K_1 \epsilon_3)] \mathbb{E}[(\|\epsilon_3\| \|G_2(\epsilon_5)\|)^2]}. \end{aligned}$$

Now,

$$\|\epsilon_5\| \leq \|MK_1 \epsilon_3\| + \|K \epsilon_4\| \leq C_{\sup} (\|\epsilon_3\| + \|\epsilon_4\|),$$

from Condition 2 ii), where, again,  $C_{\sup}$  does not depend on

$$i_1, \dots, i_{r_1}, j_1, \dots, j_{r_2}, \Delta.$$

Furthermore,  $\|K_1 \epsilon_3\| \leq C_{\sup} \|\epsilon_3\|$ . Eventually, we have

$$\begin{aligned} & |\text{Cov}(W(x_{i_1}) \dots W(x_{i_{r_1}}), W(x_{j_1}) \dots W(x_{j_{r_2}}))| \\ & \leq C_{\sup} \exp(-C_{\inf} \Delta) \sqrt{\mathbb{E}[f_1^2(K_1 \epsilon_3)]} \sqrt{\mathbb{E}\left[\left(\|\epsilon_3\| \sup_{\|x\| \leq C_{\sup} (\|\epsilon_3\| + \|\epsilon_4\|)} \|G_2(x)\|\right)^2\right]}. \end{aligned}$$

From Condition 3 i), we have  $|f_1(K_1 x)| \leq C_{\sup} e^{C_{\sup} \|x\|}$  and  $\|G_2(x)\| \leq C_{\sup} e^{C_{\sup} \|x\|}$  where  $C_{\sup}$  does not depend on  $x$  and  $i_1, \dots, i_{r_1}, j_1, \dots, j_{r_2}, \Delta$ . Hence the above square roots are finite from Lemma 6 and do not depend on  $i_1, \dots, i_{r_1}, j_1, \dots, j_{r_2}$  and  $\Delta$ . This concludes the proof.  $\square$

**Lemma 8.** Consider the setting of Section 3.1, that is  $k_Z$  satisfies Condition 2 and  $T$  satisfies Condition 3. For  $n \in \mathbb{N}$  and  $i, j = 1, \dots, n$  we have

$$\sum_{k,l=1,\dots,n} |\text{Cov}(y_i y_j, y_k y_l)| \leq C_{\sup},$$

where  $C_{\sup}$  does not depend on  $n, i, j$ .

*Proof.* We let  $d(a, (b, c)) = \min(|a - b|, |a - c|)$  for  $a, b, c \in \mathbb{R}^d$ . It is enough to show that

$$|\text{Cov}(y_i y_j, y_k y_l)| \leq C_{\sup} \exp\left(-C_{\inf} \max(d(s_k, (s_i, s_j)), d(s_l, (s_i, s_j)))\right). \quad (13)$$

Indeed, let, for  $t \geq 0$ ,  $N_{i,j,t}$  be the number of pairs  $(k, l)$ , with  $1 \leq k, l \leq n$  such that

$$t \leq \max(d(s_k, (s_i, s_j)), d(s_l, (s_i, s_j))) \leq t + 1.$$

From Condition 1, we can show that we have  $\sup_{n \in \mathbb{N}, i, j=1,\dots,n} N_{i,j,t} \leq C_{\sup} t^{2d}$ . Hence, we have

$$\begin{aligned} & \sum_{i,j=1}^n \exp\left(-C_{\inf} \max(d(s_k, (s_i, s_j)), d(s_l, (s_i, s_j)))\right) \\ & \leq \sum_{k=0}^{+\infty} C_{\sup} (k+1)^{2d} \exp(-C_{\inf} k) < +\infty. \end{aligned}$$

Thus, (13) implies the result of the lemma and it suffices to prove (13).

Let  $i, j, k, l \in \{1, \dots, n\}$  and let  $\Delta = \max(d(s_k, (s_i, s_j)), d(s_l, (s_i, s_j)))$ . By symmetry, we can consider that  $d(s_k, (s_i, s_j)) = \Delta$ .

If  $|s_k - s_l| \leq |s_i - s_l|$  and  $|s_k - s_l| \leq |s_j - s_l|$ , then  $|s_i - s_l| \geq \Delta/2$  and  $|s_j - s_l| \geq \Delta/2$ . Hence, we can apply Lemma 7 with distance  $\Delta/2$  to obtain  $|\text{Cov}(y_k y_l, y_i y_j)| \leq C_{\text{sup}} \exp(-C_{\text{inf}} \Delta/2)$ , where  $C_{\text{sup}}$  and  $C_{\text{inf}}$  do not depend on  $n, i, j, k, l, \Delta$ .

If  $|s_i - s_l| \leq |s_k - s_l|$  and  $|s_i - s_l| \leq |s_j - s_l|$ , then  $|s_k - s_l| \geq \Delta/2$ . We then have

$$\begin{aligned} \text{Cov}(y_i y_j, y_k y_l) &= \mathbb{E}[y_i y_j y_k y_l] - \mathbb{E}[y_i y_j] \mathbb{E}[y_k y_l] \\ &= \text{Cov}(y_i y_j, y_k) - \text{Cov}(y_i, y_j) \text{Cov}(y_k, y_l) \end{aligned} \quad (14)$$

since it is assumed that  $Y$  has zero-mean. In (14), the first and third covariances are bounded in absolute value by  $C_{\text{sup}} \exp(-C_{\text{inf}} \Delta/2)$  from Lemma 7, because  $|s_k - s_l| \geq \Delta/2$ ,  $|s_k - s_i| \geq \Delta/2$  and  $|s_k - s_j| \geq \Delta/2$ . Hence we have  $|\text{Cov}(y_k y_l, y_i y_j)| \leq C_{\text{sup}} \exp(-C_{\text{inf}} \Delta/2)$ , where  $C_{\text{sup}}$  and  $C_{\text{inf}}$  do not depend on  $n, i, j, k, l, \Delta$ .

If  $|s_j - s_l| \leq |s_k - s_l|$  and  $|s_j - s_l| \leq |s_i - s_l|$ , we obtain the same bound by symmetry. We have thus considered all possible cases and the proof of (13) is concluded.  $\square$

In the context of Theorem 2, the following lemma provides an approximation of  $V_n$ , based on replacing  $A$  by a sparse matrix. We remark that a similar approximation was shown in a time series context in [38]. Nevertheless, we find that our assumptions on the random field  $Y$  are more transparent and interpretable than the assumptions in [38], where cumulants are used. Because of these differences of assumptions, our proof of the following lemma differs from that in [38].

**Lemma 9.** *Let, for  $K, n \in \mathbb{N}$ ,  $A^{(K)}$  be the  $n \times n$  matrix defined by*

$$A_{i,j}^{(K)} = A_{i,j} 1_{|s_i - s_j| \leq K}.$$

*Then, under the same assumptions as in Lemma 8, we have*

$$\sup_{n \in \mathbb{N}} n \text{Var} \left( \frac{1}{n} y^\top A y - \frac{1}{n} y^\top A^{(K)} y \right) \rightarrow_{K \rightarrow \infty} 0.$$

*Proof.* For any  $K, n \in \mathbb{N}$  we have

$$\begin{aligned} &n \text{Var} \left( \frac{1}{n} y^\top A y - \frac{1}{n} y^\top A^{(K)} y \right) \\ &= \frac{1}{n} \sum_{i,j,k,l=1}^n (A - A^{(K)})_{i,j} (A - A^{(K)})_{k,l} \text{Cov}(y_i y_j, y_k y_l). \end{aligned}$$

We observe that  $|(A - A^{(K)})_{k,l}|$  is equal to 0 or is smaller than  $C_{\text{sup}}/(1 + K^{d+C_{\text{inf}}})$  by assumption. Hence we have

$$n \text{Var} \left( \frac{1}{n} y^\top A y - \frac{1}{n} y^\top A^{(K)} y \right)$$

$$\begin{aligned}
&\leq C_{\sup} \frac{1}{1 + K^{d+C_{\inf}}} \frac{1}{n} \sum_{i,j,k,l=1}^n |(A - A^{(K)})_{i,j}| |\text{Cov}(y_i y_j, y_k y_l)| \\
&\leq C_{\sup} \frac{1}{1 + K^{d+C_{\inf}}} \frac{1}{n} \sum_{i,j=1}^n |(A - A^{(K)})_{i,j}| \\
&\leq C_{\sup} \frac{1}{1 + K^{d+C_{\inf}}} \max_{i=1,\dots,n} \sum_{j=1}^n \frac{1}{1 + |s_i - s_j|^{d+C_{\inf}}} \\
&\leq C_{\sup} \frac{1}{1 + K^{d+C_{\inf}}},
\end{aligned}$$

where we have used Lemma 8 and where we have observed that  $|(A - A^{(K)})_{i,j}|$  is equal to 0 or is smaller than  $C_{\sup}/(1 + |s_i - s_j|^{d+C_{\inf}})$ . We have also used Lemma 4 in [28] for the last inequality above. All the above constants  $C_{\sup}$  and  $C_{\inf}$  naturally do not depend on  $n$ , so the lemma is proved.  $\square$

**Lemma 10.** *Consider the setting of Section 3.1, that is  $k_Z$  satisfies Condition 2 and  $T$  satisfies Condition 3. Let  $a, b \in \mathbb{N}$ . For  $i \in \{1, \dots, a\}$ , let  $\alpha_i \in \mathbb{N}$  and let  $I(i, 1), \dots, I(i, \alpha_i) \in \{1, \dots, n\}$ . For  $j \in \{1, \dots, b\}$ , let  $\beta_j \in \mathbb{N}$  and let  $J(j, 1), \dots, J(j, \beta_j) \in \{1, \dots, n\}$ . For  $i = 1, \dots, a$ , let  $f_i$  be a function from  $\mathbb{R}^{\alpha_i}$  to  $\mathbb{R}$ . For  $j = 1, \dots, b$ , let  $g_j$  be a function from  $\mathbb{R}^{\beta_j}$  to  $\mathbb{R}$ . For  $i = 1, \dots, a$ , let  $v^{(i)} = f_i(Z(s_{I(i,1)}), \dots, Z(s_{I(i,\alpha_i)}))$ . For  $j = 1, \dots, b$ , let  $w^{(j)} = g_j(Z(s_{J(j,1)}), \dots, Z(s_{J(j,\beta_j)}))$ . Let*

$$\begin{aligned}
&\alpha(\{v^{(1)}, \dots, v^{(a)}\}, \{w^{(1)}, \dots, w^{(b)}\}) = \\
&\sup \left\{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|; A \in \sigma(\{v^{(1)}, \dots, v^{(a)}\}), B \in \sigma(\{w^{(1)}, \dots, w^{(b)}\}) \right\},
\end{aligned} \tag{15}$$

where, for any set of random variables  $\{\epsilon_1, \dots, \epsilon_r\}$ ,  $\sigma(\{\epsilon_1, \dots, \epsilon_r\})$  is the sigma algebra generated by the random variables  $\{\epsilon_1, \dots, \epsilon_r\}$ . Let

$$\Delta = \inf_{\substack{i \in \{1, \dots, a\} \\ j \in \{1, \dots, b\} \\ \tilde{i} \in \{1, \dots, \alpha_i\} \\ \tilde{j} \in \{1, \dots, \beta_j\}}} |s_{I(i, \tilde{i})} - s_{J(j, \tilde{j})}|.$$

Then, we have

$$\alpha(\{v^{(1)}, \dots, v^{(a)}\}, \{w^{(1)}, \dots, w^{(b)}\}) \leq C_{\sup} e^{-C_{\inf} \Delta},$$

where  $C_{\sup}$  and  $C_{\inf}$  may depend on  $a, \alpha_1, \dots, \alpha_a$  but do not depend on  $b, (J(j, \tilde{j}))_{j=1, \dots, b, \tilde{j}=1, \dots, \beta_j}$  and  $\Delta$ .

*Proof.* Let  $\mathcal{I} = \{I(i, \tilde{i}); i = 1, \dots, a, \tilde{i} = 1, \dots, \alpha_i\}$  and let  $\mathcal{J} = \{J(j, \tilde{j}); j = 1, \dots, b, \tilde{j} = 1, \dots, \beta_j\}$ . In (15), any of the events  $A$  (resp.  $B$ ) is an event defined on the set of random variables  $\{Z(s_i)\}_{i \in \mathcal{I}}$  (resp.  $\{Z(s_j)\}_{j \in \mathcal{J}}$ ). We thus obtain

$$\alpha(\{v^{(1)}, \dots, v^{(a)}\}, \{w^{(1)}, \dots, w^{(b)}\})$$

$$\leq \alpha(\{Z(s_i)\}_{i \in \mathcal{I}}, \{Z(s_j)\}_{j \in \mathcal{J}}) \\ := \sup \left\{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|; A \in \sigma(\{Z(s_i)\}_{i \in \mathcal{I}}), B \in \sigma(\{Z(s_j)\}_{j \in \mathcal{J}}) \right\}.$$

Let  $I_1 < \dots < I_{\bar{\alpha}}$  be such that  $\{I_1, \dots, I_{\bar{\alpha}}\} = \mathcal{I}$  and let

$$v_Z = (Z(s_{I_1}), \dots, Z(s_{I_{\bar{\alpha}}}))^\top.$$

Let  $J_1 < \dots < J_{\bar{\beta}}$  be such that  $\{J_1, \dots, J_{\bar{\beta}}\} = \mathcal{J}$  and let

$$w_Z = (Z(s_{J_1}), \dots, Z(s_{J_{\bar{\beta}}}))^\top.$$

From Lemma 1 in Section 2.1 of [23], we have

$$\alpha(\{Z(s_i)\}_{i \in \mathcal{I}}, \{Z(s_j)\}_{j \in \mathcal{J}}) \\ \leq \sup \left\{ |\text{Cov}(v^\top v_Z, w^\top w_Z)|; \text{Var}(v^\top v_Z) = 1, \text{Var}(w^\top w_Z) = 1 \right\}. \quad (16)$$

Let  $v$  and  $w$  be vectors belonging to the set in (16). The smallest eigenvalues of the covariance matrices of  $v_Z$  and  $w_Z$  are larger than a constant  $C_{\inf}$ , not depending on  $\mathcal{I}$  and  $\mathcal{J}$ , since  $k_Z$  satisfies Condition 2. Thus we have

$$1 = \text{Var}(v^\top v_Z) = v^\top \text{Cov}(v_Z) v \geq C_{\inf} \|v\|^2.$$

It follows that  $\|v\|^2 \leq C_{\sup}$ , where  $C_{\sup}$  does not depend on  $\mathcal{I}$ ,  $\mathcal{J}$  and  $\Delta$ . Similarly  $\|w\|^2 \leq C_{\sup}$ .

We have

$$\begin{aligned} \text{Cov}^2(v^\top v_Z, w^\top w_Z) &\leq \|\text{Cov}(v^\top v_Z, w_Z)\|^2 \|w\|^2 \\ &\leq C_{\sup} \sum_{j=1, \dots, \bar{\beta}} \text{Cov}(v^\top v_Z, Z(s_{J_j}))^2 \\ &\leq C_{\sup} \|v\|^2 \sum_{i=1, \dots, \bar{\alpha}} \sum_{j=1, \dots, \bar{\beta}} \text{Cov}(Z(s_{I_i}), Z(s_{J_j}))^2 \\ &\leq C_{\sup} \sum_{i=1, \dots, \bar{\alpha}} \sum_{\substack{j=1, \dots, n \\ |s_j - s_i| \geq \Delta}} e^{-C_{\inf} |s_{I_i} - s_j|}, \end{aligned}$$

by definition of  $\Delta$ , since  $k_Z$  satisfies Condition 2 and where  $C_{\sup}$  and  $C_{\inf}$  do not depend on  $b$ ,  $\mathcal{J}$  and  $\Delta$ . For any  $i \in \{1, \dots, n\}$ , the number of indices  $j \in \{1, \dots, n\}$  such that  $\tilde{\Delta} \leq |s_i - s_j| \leq \tilde{\Delta} + 1$  is smaller than  $C_{\sup} \tilde{\Delta}^d$ , from Condition 1 and where  $C_{\sup}$  only depends on  $d$ . This yields

$$\begin{aligned} \text{Cov}^2(v^\top v_Z, w^\top w_Z) &\leq C_{\sup} (\alpha_1 + \dots + \alpha_a) \sum_{k=0}^{+\infty} C_{\sup} (\Delta + k)^d e^{-C_{\inf} |\Delta + k|} \\ &\leq C_{\sup} e^{-C_{\inf} |\Delta|/2} (\alpha_1 + \dots + \alpha_a) \sum_{k=1}^{+\infty} (\Delta + k)^d e^{-C_{\inf} |\Delta + k|/2} \end{aligned}$$

$$\leq C_{\sup} e^{-C_{\inf}|\Delta|}$$

where the different  $C_{\sup}$  and  $C_{\inf}$  do not depend on  $b$ ,  $\mathcal{J}$  and  $\Delta$ . This concludes the proof from (16).  $\square$

**Lemma 11.** *Consider a sequence  $(x_i)_{i \in \mathbb{N}}$  of points in  $\mathbb{R}^d$  satisfying Condition 1. Let  $\tau > 0$  be fixed. For  $n \in \mathbb{N}$ , let  $(A_\theta)_{\theta \in \Theta}$  and  $(B_\theta)_{\theta \in \Theta}$  be families of  $n \times n$  matrices. Assume that for all  $n \in \mathbb{N}$ ,  $i, j = 1, \dots, n$  and  $\theta \in \Theta$ ,*

$$|(A_\theta)_{i,j}| \leq \frac{C_{\sup}}{1 + |x_i - x_j|^{d+\tau}} \quad \text{and} \quad |(B_\theta)_{i,j}| \leq \frac{C_{\sup}}{1 + |x_i - x_j|^{d+\tau}}$$

where  $C_{\sup}$  does not depend on  $n, i, j, \theta$ . Then we have for all  $n \in \mathbb{N}$ ,  $i, j = 1, \dots, n$  and  $\theta \in \Theta$ ,

$$|(A_\theta B_\theta)_{i,j}| \leq \frac{C_{\sup}}{1 + |x_i - x_j|^{d+\tau}}$$

where  $C_{\sup}$  does not depend on  $n, i, j, \theta$ .

*Proof.* We have,

$$\begin{aligned} |(A_\theta B_\theta)_{i,j}| &= \left| \sum_{\ell=1}^n (A_\theta)_{i,\ell} (B_\theta)_{\ell,j} \right| \\ &\leq \sum_{\ell=1}^n \frac{C_{\sup}}{1 + |x_i - x_\ell|^{d+\tau}} \frac{C_{\sup}}{1 + |x_j - x_\ell|^{d+\tau}} \\ &\leq \sum_{\substack{\ell=1, \dots, n \\ |x_i - x_\ell| \leq |x_j - x_\ell|}} \frac{C_{\sup}}{1 + |x_i - x_\ell|^{d+\tau}} \frac{C_{\sup}}{1 + (|x_i - x_j|/2)^{d+\tau}} \\ &\quad + \sum_{\substack{\ell=1, \dots, n \\ |x_j - x_\ell| \leq |x_i - x_\ell|}} \frac{C_{\sup}}{1 + (|x_i - x_j|/2)^{d+\tau}} \frac{C_{\sup}}{1 + |x_j - x_\ell|^{d+\tau}} \\ &\leq C_{\sup} \frac{C_{\sup}}{1 + |x_i - x_j|^{d+\tau}} \max_{a=1, \dots, n} \sum_{b=1}^n \frac{1}{1 + |x_a - x_b|^{d+\tau}} \\ &\leq \frac{C_{\sup}}{1 + |x_i - x_j|^{d+\tau}} \end{aligned}$$

from Lemma 4 in [28].  $\square$

**Lemma 12.** *Consider the setting of Section 5.1. Under Conditions 4 and 5, we have*

$$\sup_{\theta \in \Theta} \lambda_1(R_\theta^{-1}) \leq C_{\sup}, \quad (17)$$

$$\sup_{\theta \in \Theta} \lambda_1(R_\theta) \leq C_{\sup}, \quad (18)$$

and

$$\sup_{\substack{\theta \in \Theta \\ \ell=1,2,3 \\ i_1, \dots, i_\ell=1, \dots, p}} \lambda_1 \left( \frac{\partial R_\theta}{\partial \theta_{i_1} \dots \partial \theta_{i_\ell}} \right) \leq C_{\sup}. \quad (19)$$

*Proof.* Conditions 1, 4 and 5 imply (17) from Theorem 5 in [12]. Conditions 1 and 4 imply (18) and (19) from Lemma 6 in [28].  $\square$

**Lemma 13.** Consider the setting of Section 5.1. Under Conditions 4 and 5, we have, for  $n \in \mathbb{N}$  and  $i, j \in \{1, \dots, n\}$ ,

$$\sup_{\theta \in \Theta} |(R_\theta^{-1})_{i,j}| \leq \frac{C_{\sup}}{1 + |s_i - s_j|^{d+C_{\inf}}},$$

where  $C_{\sup}$  and  $C_{\inf}$  do not depend on  $n, i, j, \theta$ .

*Proof.* One can show that the proof of Theorem 1 can be made uniform over  $\theta \in \Theta$ , thus yielding Lemma 13.  $\square$

**Lemma 14.** Consider the setting of Section 5.1. Under Conditions 4 and 5, we have,

$$\inf_{\theta \in \Theta} \lambda_n(R_\theta) \geq C_{\inf}, \quad (20)$$

$$\inf_{\theta \in \Theta} \lambda_n(\text{diag}(R_\theta^{-1})) \geq C_{\inf}. \quad (21)$$

*Proof.* Equation (20) holds from (17). Then, (21) follows from (20) as in Lemma D.6 in [7].  $\square$

## A.2. Proofs of the main results

**Proof of Lemma 1.** As a special case of Lemma 7,  $k'$  satisfies Condition 2 i).

Let us now show that  $k'$  satisfies Condition 2 ii). Let  $(x_i)$  satisfy Condition 1. Let  $n \in \mathbb{N}$  be fixed and let  $R$  be the  $n \times n$  covariance matrix  $k(x_i - x_j)_{i,j=1, \dots, n}$ . Let  $a_1, \dots, a_n \in \mathbb{R}$ . We have

$$\sum_{i,j=1}^n a_i a_j R_{i,j} = \text{Var} \left( \sum_{i=1}^n a_i F(X(x_i)) \right).$$

We now let  $z = (X(x_1), \dots, X(x_n))^\top$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined by  $g(t) = \sum_{i=1}^n a_i F(t_i)$ . The gradient of  $g$  at  $t$  is  $\nabla g(t) = (a_1 F'(t_1), \dots, a_n F'(t_n))^\top$ . We use the inequality in Theorem 3.7 in [19]. This yields

$$\sum_{i,j=1}^n a_i a_j R_{i,j} \geq \mathbb{E} [\nabla g(z)]^\top \text{Cov}(z) \mathbb{E} [\nabla g(z)].$$

From Condition 2 ii), we have  $\lambda_1(\text{Cov}(z)) \geq C_{\inf}$ . This yields

$$\sum_{i,j=1}^n a_i a_j R_{i,j} \geq C_{\inf} \sum_{i=1}^n \mathbb{E}^2 [(\nabla g(z))_i]$$

$$\begin{aligned}
&= C_{\inf} \sum_{i=1}^n a_i^2 \mathbb{E}[F'(z_i)]^2 \\
&= C_{\inf} \left( \sum_{i=1}^n a_i^2 \right) \mathbb{E}[F'(z_1)]^2.
\end{aligned}$$

From Condition 3, the above expectation is non-zero, which concludes the proof.  $\square$

**Proof of Lemma 2.** The fact that  $k$  satisfies Condition 2 ii) follows from Theorem 4 in [12].

Let us now consider the case where  $F$  is defined by  $F(x) = x^{2r} + u$ . Since we consider a covariance function we can assume that  $u = 0$  without loss of generality. Assume also that  $k(0) = 1$  without loss of generality. From Lemma 7,  $k'$  satisfies Condition 2 i). Let us show that Condition 2 ii) is satisfied. Let  $a, b \in \mathbb{R}^d$ , let  $c = k(a - b)$  and let  $\lambda = (1 - c^2)^{1/2}$ . With  $(A_1, A_2) \sim \mathcal{N}[0, I_2]$ , we have

$$\begin{aligned}
k'(a - b) &= \text{Cov}(A_1^{2r}, (cA_1 + \lambda A_2)^{2r}) \\
&= \text{Cov}\left(A_1^{2r}, \sum_{i=0}^{2r} \binom{2r}{i} c^i \lambda^{2r-i} A_1^i A_2^{2r-i}\right) \\
&= \sum_{i=0}^{2r} \binom{2r}{i} c^i \lambda^{2r-i} \text{Cov}(A_1^{2r}, A_1^i A_2^{2r-i}).
\end{aligned}$$

By independence of  $A_1$  and  $A_2$ , we obtain, for  $i = 0, \dots, 2r$ ,

$$\text{Cov}(A_1^{2r}, A_1^i A_2^{2r-i}) = \mathbb{E}[A_2^{2r-i}] (\mathbb{E}[A_1^{2r+i}] - \mathbb{E}[A_1^{2r}] \mathbb{E}[A_1^i]). \quad (22)$$

From Isserlis' theorem, one can show that (22) is zero if  $i$  is odd and is strictly positive if  $i$  is even. As a consequence, we have

$$k'(a - b) = \sum_{i=0}^r \alpha_i k(a - b)^{2i}$$

with  $\alpha_1, \dots, \alpha_r > 0$ . Hence, the Fourier transform of  $k'$  is a linear combination of multiple convolutions of the Fourier transform of  $k$ , with strictly positive components. Since the Fourier transform of  $k$  is strictly positive everywhere, then also the Fourier transform of  $k'$  is strictly positive everywhere. Hence, from Theorem 4 in [12],  $k'$  satisfies Condition 2 ii).  $\square$

**Proof of Theorem 1.** Condition 1 and Lemma 6 in [28] imply that the spectral norms of  $R^{-1}$  and  $R$  are bounded functions of  $n$ . Let  $C_{\sup} = \sup_n \lambda_1(R) < \infty$  and  $C_{\inf} = \inf_n \lambda_n(R) > 0$ .

We write

$$R^{-1} = \frac{1}{C_{\sup}} \left( I - \left( I - \frac{R}{C_{\sup}} \right) \right)^{-1} = \frac{1}{C_{\sup}} \sum_{\ell=0}^{\infty} \left( I - \frac{R}{C_{\sup}} \right)^{\ell}. \quad (23)$$



We remark that the above sum is well-defined because the eigenvalues of  $I - R/C_{\text{sup}}$  are between 0 and  $1 - C_{\text{inf}}/C_{\text{sup}}$ .

We denote  $M = I - C_{\text{sup}}^{-1}R$  and  $h_{i,j} = |s_i - s_j|$ . Let  $1 \leq A < \infty$  and  $a > 0$  be fixed such that  $M_{i,j} \leq Ae^{-2ah_{i,j}}$ . Let  $\delta = \inf_{i,j \in \mathbb{N}, i \neq j} |s_i - s_j| > 0$  (Condition 1). Let  $D < \infty$  be a constant such that  $D \geq 1$  and, for any  $L \geq \delta$  and  $i \in \mathbb{N}$ , the set  $\{j \in \mathbb{N}; |s_i - s_j| \leq L\}$  has no more than  $(D/2)L^d$  elements.

Let  $0 < \mu < \infty$  be fixed. We show by induction over  $\ell \in \mathbb{N}$  that there exists a constant  $1 \leq \varphi < \infty$ , depending on  $\mu$  but not depending on  $\ell, i, j$ , such that for  $\ell \leq \mu \log(h_{i,j})$ ,

$$|(M^\ell)_{i,j}| \leq A^\ell \varphi^\ell D^\ell h_{i,j}^{d\ell-d} e^{-ah_{i,j}}. \quad (24)$$

In the case  $\log(h_{i,j}) < 0$ , there is nothing to prove in (24), so we consider  $i, j$  such that  $\log(h_{i,j}) \geq 0$  when proving (24).

For  $\ell = 1$ , (24) holds. Assume that (24) holds for some  $\ell \in \mathbb{N}$ . We have

$$\begin{aligned} |(M^{\ell+1})_{i,j}| &= \sum_{r=1}^n (M^\ell)_{i,r} M_{j,r} \\ &\leq \sum_{r=1}^n A^\ell \varphi^\ell D^\ell h_{i,r}^{d\ell-d} e^{-ah_{i,r}} A e^{-2ah_{r,j}} \\ &= A^{\ell+1} \varphi^\ell D^\ell \sum_{r=1}^n h_{i,r}^{d\ell-d} e^{-ah_{i,r}} e^{-2ah_{r,j}}. \end{aligned}$$

Let now  $B_i = \{x \in \mathbb{R}^d; |x - s_i| \leq |s_i - s_j|\}$  and  $B_j = \{x \in \mathbb{R}^d; |x - s_j| \leq |s_i - s_j|\}$ . From the triangle inequality we obtain

$$\begin{aligned} |(M^{\ell+1})_{i,j}| &\leq A^{\ell+1} \varphi^\ell D^\ell \sum_{r \in \mathbb{N}; s_r \in B_i \cup B_j} h_{i,r}^{d\ell-d} e^{-ah_{i,r}} e^{-2ah_{r,j}} \\ &\quad + A^{\ell+1} \varphi^\ell D^\ell \sum_{r \in \mathbb{N}; s_r \in B_i^c \cap B_j^c} h_{i,r}^{d\ell-d} e^{-ah_{i,r}} e^{-2ah_{r,j}} \\ &\leq A^{\ell+1} \varphi^\ell D^\ell 2(D/2) h_{i,j}^d h_{i,j}^{d\ell-d} e^{-ah_{i,j}} \\ &\quad + A^{\ell+1} \varphi^\ell D^\ell e^{-2ah_{i,j}} \sum_{r \in \mathbb{N}; s_r \in B_i^c} h_{i,r}^{d\ell-d} e^{-ah_{i,r}} \\ &\leq A^{\ell+1} \varphi^\ell D^{\ell+1} h_{i,j}^{d(\ell+1)-d} e^{-ah_{i,j}} \\ &\quad + A^{\ell+1} \varphi^\ell D^\ell e^{-ah_{i,j}} \left( e^{-ah_{i,j}} \sum_{r \in \mathbb{N}} Q r^{d\mu \log(h_{i,j})} e^{-a(r-1)} \right), \end{aligned}$$

where for the last inequality we let  $Qr^d$  be an upper bound on the cardinality of  $\{b \in \mathbb{N}; |s_b - s_i| \in [r-1, r]\}$  for all  $i \in \mathbb{N}$ . The constant  $Q$  is finite and depends only on  $d$  and  $\delta$  from Condition 1. We also let  $\ell \leq \mu \log(h_{i,j})$  to show the last above inequality. Hence, in order to finish the proof of (24), it remains to show that the term  $(.)$  in the above display is a bounded function of  $h_{i,j}$ , and to let  $\varphi/2 \geq 1$  be a bound for the term  $(.)$ .

We have, for  $h_{i,j}$  large enough, with  $\lceil \cdot \rceil$  the integer ceiling,

$$\begin{aligned}
& \left( e^{-ah_{i,j}} \sum_{r \in \mathbb{N}} Q r^{d\mu \log(h_{i,j})} e^{-a(r-1)} \right) \\
&= Q e^a e^{-ah_{i,j}} \sum_{r \in \mathbb{N}} r^{\lceil d\mu \log(h_{i,j}) \rceil} e^{-ar} \\
&= Q e^a e^{-ah_{i,j}} \left( \frac{2}{a} \right)^{\lceil d\mu \log(h_{i,j}) \rceil} \sum_{r \in \mathbb{N}} \left( \frac{ar}{2} \right)^{\lceil d\mu \log(h_{i,j}) \rceil} e^{-ar} \\
&\leq Q e^a e^{-ah_{i,j}} \left( \frac{2}{a} \right)^{\lceil d\mu \log(h_{i,j}) \rceil} \lceil d\mu \log(h_{i,j}) \rceil! \sum_{r \in \mathbb{N}} e^{ar/2} e^{-ar} \\
&= Q e^a e^{-ah_{i,j}} \left( \frac{2}{a} \right)^{\lceil d\mu \log(h_{i,j}) \rceil} \lceil d\mu \log(h_{i,j}) \rceil! \frac{1}{1 - e^{-ar/2}}.
\end{aligned}$$

The above function of  $h_{i,j}$  clearly goes to 0 as  $h_{i,j}$  goes to  $\infty$ . Thus, the above term  $(\cdot)$  is bounded and thus (24) is proved.

Coming back to (23), using (24) and using the triangle inequality, we obtain, letting  $\Delta = 1 - C_{\inf}/C_{\sup}$ , and for  $h_{i,j}$  large enough,

$$\begin{aligned}
|(R^{-1})_{i,j}| &\leq \left( \sum_{1 \leq \ell \leq \mu \log(h_{i,j})} A^\ell \varphi^\ell D^\ell h_{i,j}^{d\ell-d} e^{-ah_{i,j}} \right) + \sum_{\mu \log(h_{i,j}) \leq \ell \leq \infty} \Delta^\ell \\
&\leq \mu \log(h_{i,j}) (A\varphi D)^{\mu \log(h_{i,j})} h_{i,j}^{d\mu \log(h_{i,j})} e^{-ah_{i,j}} + \frac{\Delta^{\mu \log(h_{i,j})}}{1 - \Delta} \\
&= \mu \log(h_{i,j}) (A\varphi D)^{\mu \log(h_{i,j})} h_{i,j}^{d\mu \log(h_{i,j})} e^{-ah_{i,j}} + \frac{h_{i,j}^{\mu \log(\Delta)}}{1 - \Delta}. \quad (25)
\end{aligned}$$

In the above display, for any  $\tau < \infty$  in the statement of Theorem 1, we can choose  $\mu$  such that  $\mu \log(\Delta) \leq -d - \tau$ . Then, it is clear that the first summand in (25) is also smaller than a constant (depending on  $\tau$ ) time  $h_{i,j}^{-d-\tau}$ . This concludes the proof of Theorem 1, since also  $\sup_{n \in \mathbb{N}} \max_{i,j=1,\dots,n} |(R^{-1})_{i,j}|$  is bounded by  $C_{\sup}$ .  $\square$

**Proof of Theorem 2.** We have

$$n \operatorname{Var}(V_n) = \frac{1}{n} \sum_{i,j,k,l=1}^n A_{i,j} A_{k,l} \operatorname{Cov}(y_i y_j, y_k y_l). \quad (26)$$

From Lemma 8, we obtain

$$\begin{aligned}
n \operatorname{Var}(V_n) &\leq \frac{1}{n} C_{\sup} \sum_{i,j=1}^n |A_{i,j}| \\
&\leq C_{\sup} \max_{i=1,\dots,n} \sum_{j=1}^n \frac{1}{1 + |s_i - s_j|^{d+C_{\inf}}} \\
&\leq C_{\sup}
\end{aligned}$$

from Lemma 4 in [28]. Hence,  $n \text{Var}(V_n)$  is bounded as  $n \rightarrow \infty$ .

Assume now that

$$d_w(\mathcal{L}_n, \mathcal{N}[0, n \text{Var}(V_n)]) \not\rightarrow 0 \quad (27)$$

as  $n \rightarrow \infty$ . Because  $n \text{Var}(V_n)$  is bounded, there exists a subsequence  $\phi(n)$  such that

$$d_w(\mathcal{L}_{\phi(n)}, \mathcal{N}[0, \phi(n) \text{Var}(V_{\phi(n)})]) \not\rightarrow 0 \quad (28)$$

as  $n \rightarrow \infty$  and  $\phi(n) \text{Var}(V_{\phi(n)}) \rightarrow V \in [0, \infty)$  as  $n \rightarrow \infty$ . It is then simple to show that this implies

$$d_w(\mathcal{L}_{\phi(n)}, \mathcal{N}[0, V]) \not\rightarrow 0 \quad (29)$$

as  $n \rightarrow \infty$ . If  $V = 0$ , then, from Chebyshev inequality,  $\mathcal{L}_{\phi(n)}$  converges to a Dirac mass at zero and so (29) does not hold, yielding a contradiction.

Hence it remains to consider the case  $\phi(n) \text{Var}(V_{\phi(n)}) \rightarrow V \in (0, \infty)$  as  $n \rightarrow \infty$  and where (29) holds.

To reach a contradiction, we will show that

$$\sqrt{\phi(n)} \left( \frac{V_{\phi(n)} - \mathbb{E}[V_{\phi(n)}]}{\sqrt{V}} \right) \xrightarrow{\mathcal{L}_{n \rightarrow \infty}} \mathcal{N}[0, 1].$$

To simplify notations in the sequel, without loss of generality, we will consider that  $\phi(n) = n$  and show that

$$\sqrt{n} \left( \frac{V_n - \mathbb{E}[V_n]}{\sqrt{V}} \right) \xrightarrow{\mathcal{L}_{n \rightarrow \infty}} \mathcal{N}[0, 1], \quad (30)$$

where  $n \text{Var}(V_n) \rightarrow V \in (0, \infty)$  as  $n \rightarrow \infty$ . From Slutsky's lemma it is sufficient to show that

$$\sqrt{n} \left( \frac{V_n - \mathbb{E}[V_n]}{\sqrt{n \text{Var}(V_n)}} \right) \xrightarrow{\mathcal{L}_{n \rightarrow \infty}} \mathcal{N}[0, 1]. \quad (31)$$

For  $K \geq 0$ , let

$$V_n^{(K)} = \frac{1}{n} y^\top A^{(K)} y,$$

with the notation of Lemma 9. We have

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left( \sqrt{n} \left( \frac{V_n - \mathbb{E}[V_n]}{\sqrt{n \text{Var}(V_n)}} \right) - \sqrt{n} \left( \frac{V_n^{(K)} - \mathbb{E}[V_n^{(K)}]}{\sqrt{n \text{Var}(V_n^{(K)})}} \right) \right)^2 \right] \\ & \leq 2 \sup_{n \in \mathbb{N}} \text{Var} \left( \sqrt{n} \left( \frac{V_n - V_n^{(K)}}{\sqrt{n \text{Var}(V_n)}} \right) \right) \\ & \quad + 2 \sup_{n \in \mathbb{N}} \text{Var} \left( \sqrt{n} V_n^{(K)} \left( \frac{1}{\sqrt{n \text{Var}(V_n)}} - \frac{1}{\sqrt{n \text{Var}(V_n^{(K)})}} \right) \right) \\ & \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

from Lemma 9 and because  $V > 0$ . Hence, from Theorem 4.2 in [18] (as in [38]), it is sufficient to show that there exists  $L \in (0, \infty)$  such that for any fixed  $K \geq L$ , we have

$$\sqrt{n} \left( \frac{V_n^{(K)} - \mathbb{E}[V_n^{(K)}]}{\sqrt{n \operatorname{Var}(V_n^{(K)})}} \right) \xrightarrow{\mathcal{L}} \mathcal{N}[0, 1]$$

as  $n \rightarrow \infty$ , in order to prove (31) and thus to conclude the proof. We remark that, because of Lemma 9, we have  $|\liminf_{n \rightarrow \infty} n \operatorname{Var}(V_n^{(K)}) - \liminf_{n \rightarrow \infty} n \operatorname{Var}(V_n)|$  goes to 0 as  $K \rightarrow \infty$ . Hence, we may take  $L$  such that  $\liminf_{n \rightarrow \infty} n \operatorname{Var}(V_n^{(K)}) > 0$  for  $K \geq L$ . Hence, up to extracting a subsequence, it is sufficient to show

$$\sqrt{n}(V_n^{(K)} - \mathbb{E}[V_n^{(K)}]) \xrightarrow{\mathcal{L}} \mathcal{N}[0, V^{(K)}], \quad (32)$$

where  $n \operatorname{Var}(V_n^{(K)}) \rightarrow V^{(K)} > 0$  as  $n \rightarrow \infty$ . We have

$$\begin{aligned} & \sqrt{n}(V_n^{(K)} - \mathbb{E}[V_n^{(K)}]) \\ &= \frac{1}{\sqrt{n}} \sum_{i,j=1}^n (y_i y_j - \mathbb{E}[y_i y_j]) A_{i,j} 1_{|s_i - s_j| \leq K} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \sum_{\substack{j=1, \dots, n \\ |s_i - s_j| \leq K}} A_{i,j} (T(Z(s_i))T(Z(s_j)) - \mathbb{E}[T(Z(s_i))T(Z(s_j))]) \right) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n X_n(s_i), \end{aligned}$$

say, where  $X_n$  can be interpreted as a centered random field defined on  $(s_i)_{i \in \mathbb{N}}$ . We will now show that the sequence of random fields  $(X_n)_{n \in \mathbb{N}}$  satisfies the conditions of Corollary 1 of [32].

We let, for  $k, l \in \mathbb{N}$  and  $r \geq 0$

$$\begin{aligned} \bar{\alpha}_{k,l}(r) &= \sup_{n \in \mathbb{N}} \sup \left\{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|; \right. \\ & \quad A \in \sigma(X_n(s_{I_1}), \dots, X_n(s_{I_{\bar{k}}}), B \in \sigma(X_n(s_{J_1}), \dots, X_n(s_{J_{\bar{l}})}), \\ & \quad \bar{k} \leq k, \bar{l} \leq l, I_1, \dots, I_{\bar{k}}, J_1, \dots, J_{\bar{l}} \in \{1, \dots, n\}, \\ & \quad \left. \min_{\substack{\bar{k}=1, \dots, \bar{k}, \bar{l}=1, \dots, \bar{l}}} |s_{I_{\bar{k}}} - s_{J_{\bar{l}}}| \geq r \right\}. \end{aligned}$$

Let  $N_K = \sup_{n \in \mathbb{N}} \max_{i=1, \dots, n} \sum_{j=1}^n 1_{|s_j - s_i| \leq K}$ . Then  $N_K \leq C_{\sup}$ , where  $C_{\sup}$  depends only on  $K$ ,  $d$  and  $\delta$  from Condition 1. We remark that for  $i = 1, \dots, n$ ,  $X_n(s_i)$  is a function of the variables  $Z(s_{I(n,i,1)}), \dots, Z(s_{I(n,i,\gamma(n,i))})$ , with  $\gamma(n,i) \leq N_K$  and with  $|s_{I(n,i,\gamma)} - s_i| \leq K$  for  $\gamma = 1, \dots, \gamma(n,i)$ . Furthermore, for  $|s_i - s_j| \geq r$  we have for  $\gamma_i = 1, \dots, \gamma(n,i)$  and for  $\gamma_j = 1, \dots, \gamma(n,j)$  that  $|s_{I(n,i,\gamma_i)} - s_{I(n,j,\gamma_j)}| \geq r - 2K$ . Hence, from Lemma 10, we have, for any  $k \in \mathbb{N}$

$$\sup_{l \in \mathbb{N}} \bar{\alpha}_{k,l}(r) \leq C_{\sup} e^{-C_{\inf} r},$$

where  $C_{\sup}$  and  $C_{\inf}$  may depend on  $k$  and  $K$ .

We now let  $D = (s_i)_{i \in \mathbb{N}}$ ,  $D_n = (s_1, \dots, s_n)$ ,  $Z_{i,n} = n^{-1/2}X_n(s_i)$  for  $n \in \mathbb{N}$  and  $i = 1, \dots, n$ . We also let  $c_{i,n} = n^{-1/2}$  for  $n \in \mathbb{N}$  and  $i = 1, \dots, n$ . We remark that  $Z_{i,n}/c_{i,n}$  can be written as  $f(w)$  where  $w$  is a Gaussian vector of dimension less than  $N_K$ , with variances 1 and where  $|f(x)| \leq C_{\sup}e^{C_{\sup}|x|}$ , where  $C_{\sup}$  does not depend on  $n \in \mathbb{N}$  and  $i = 1, \dots, n$ . One can thus show, from the Cauchy-Schwarz inequality and with the same techniques as in Lemma 6, that for any  $q > 0$

$$\lim_{M \rightarrow +\infty} \sup_{n \in \mathbb{N}} \max_{i=1, \dots, n} \mathbb{E} [|Z_{i,n}/c_{i,n}|^{2+q} 1_{|Z_{i,n}/c_{i,n}| \geq M}] = 0. \quad (33)$$

With the previous notation and with (33), one can show that all the assumptions of Corollary 1 in [32] are satisfied. This shows (32) and thus concludes the proof.  $\square$

**Proof of Theorem 3.** Let  $\theta \in \Theta$  be fixed. We have

$$\text{Var}(L_\theta) = \frac{1}{n} \text{Var} \left( \frac{1}{n} (y^\top R_\theta^{-1} y) \right).$$

From Lemma 13 and Theorem 2, applied with  $A_n = R_\theta^{-1}$ , we obtain  $\text{Var}(L_\theta) \rightarrow 0$  as  $n \rightarrow \infty$ . For  $i = 1, \dots, p$ ,

$$\frac{\partial L_\theta}{\partial \theta_i} = \frac{1}{n} \text{tr} \left( R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_i} \right) + \frac{1}{n} \left( y^\top \left( -R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_i} R_\theta^{-1} \right) y \right),$$

which can be rewritten for convenience as

$$\frac{\partial L_\theta}{\partial \theta_i} = \frac{1}{n} \text{tr} (P_{\theta,i}) + \frac{1}{n} (y^\top Q_{\theta,i} y)$$

with

$$P_{\theta,i} = R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_i} \quad \text{and} \quad Q_{\theta,i} = -R_\theta^{-1} \frac{\partial R_\theta}{\partial \theta_i} R_\theta^{-1}.$$

The matrices  $R_\theta^{-1}$  and  $\partial R_\theta / \partial \theta_i$  are both valid choices for  $A_\theta$  and  $B_\theta$  in Lemma 11. From Gerschgorin Circle Theorem (GCT) and Lemma 4 in [28], we obtain  $\sup_{\theta \in \Theta} \lambda_1(P_{\theta,i}^\top P_{\theta,i}) \leq C_{\sup}$  and  $\sup_{\theta \in \Theta} \lambda_1(Q_{\theta,i}) \leq C_{\sup}$ . This, in turn implies that  $\sup_{\theta \in \Theta} \rho_1(P_{\theta,i}) \leq C_{\sup}$ . It follows that

$$\begin{aligned} \max_{i=1, \dots, n} \sup_{\theta \in \Theta} \left| \frac{\partial L_\theta}{\partial \theta_i} \right| &\leq \sup_{\theta \in \Theta} \left| \frac{1}{n} \text{tr} (P_{\theta,i}) + \frac{1}{n} (y^\top Q_{\theta,i} y) \right| \\ &\leq C_{\sup} + C_{\sup} \frac{\|y\|^2}{n} \\ &= O_p(1). \end{aligned}$$

Hence, Theorem 3 can be proved by proceeding as in the proof of Proposition 3.1 in [7].  $\square$

**Proof of Theorem 4.** From the proof of Theorem 3, we have for  $i = 1, \dots, p$

$$\partial L_\theta / \partial \theta_i = \frac{1}{n} \text{tr}(P_{\theta,i}) + \frac{1}{n} (y^\top Q_{\theta,i} y),$$

where  $P_{\theta,i}$  is a  $n \times n$  matrix satisfying  $\sup_{\theta \in \Theta} |(P_{\theta,i})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$  and  $Q_{\theta,i}$  is a  $n \times n$  symmetric matrix satisfying  $\sup_{\theta \in \Theta} |(Q_{\theta,i})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$ .

One can check that  $\partial L_{\theta_0} / \partial \theta_i$  has mean zero for  $i = 1, \dots, p$ , since the mean value of  $\partial L_{\theta_0} / \partial \theta_i$  is calculated as if  $Y$  were a Gaussian process with zero-mean and covariance function  $k_{Y,\theta_0}$ . Let  $\partial L_{\theta_0} / \partial \theta$  be the gradient column vector of  $L_\theta$  at  $\theta_0$ . From Theorem 2, with  $\mathcal{L}_{\Sigma,\theta_0,n}$  the distribution of  $\sqrt{n} \partial L_{\theta_0} / \partial \theta$ , as  $n \rightarrow \infty$ ,

$$d_w(\mathcal{L}_{\Sigma,\theta_0,n}, \mathcal{N}[0, \Sigma_{\theta_0}]) \rightarrow 0. \quad (34)$$

In addition, for  $i \in \{1, \dots, p\}$ , the sequence  $(n \text{Var}(\partial L_{\theta_0} / \partial \theta_i))$  is bounded, which implies that the elements of  $\Sigma_{\theta_0}$  are bounded too.

One can check that the mean value of  $\partial L_{\theta_0} / \partial \theta_i \partial \theta_j$  is  $(M_{\theta_0})_{i,j}$  (also because this mean value is calculated as if  $Y$  were a Gaussian process with zero-mean and covariance function  $k_{Y,\theta_0}$ ).

Also, for  $i, j = 1, \dots, p$ , we have

$$\partial L_\theta / \partial \theta_i \partial \theta_j = \frac{1}{n} \text{tr}(C_{\theta,i,j}) + \frac{1}{n} (y^\top D_{\theta,i,j} y),$$

where  $C_{\theta,i,j}$  and  $D_{\theta,i,j}$  are sums of products of the matrices  $R_\theta^{-1}$ ,  $R_\theta$  and the first and second derivative matrices of  $R_\theta$  (see e.g., [7]). Hence, from Condition 4 and Lemma 13 used inside Lemma 11, we have  $\sup_{\theta \in \Theta} |(C_{\theta,i,j})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$  and  $\sup_{\theta \in \Theta} |(D_{\theta,i,j})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$ .

Thus, the variance of  $\partial L_{\theta_0} / \partial \theta_i \partial \theta_j$  goes to zero as  $n \rightarrow \infty$  from Theorem 2. Hence

$$\partial L_{\theta_0} / \partial \theta_i \partial \theta_j \rightarrow^p (M_{\theta_0})_{i,j} \quad (35)$$

as  $n \rightarrow \infty$ .

It can be shown, similarly as in the proof of Proposition 3.3 in [7] that

$$\liminf_{n \rightarrow \infty} \lambda_p(M_{\theta_0}) > 0. \quad (36)$$

Hence, (7) follows.

Then, for  $i, j, \ell \in \{1, \dots, p\}$ , we have

$$\partial L_\theta / \partial \theta_i \partial \theta_j \partial \theta_\ell = \frac{1}{n} \text{tr}(E_{\theta,i,j,\ell}) + \frac{1}{n} (y^\top F_{\theta,i,j,\ell} y),$$

where  $E_{\theta,i,j,\ell}$  and  $F_{\theta,i,j,\ell}$  are sums of products of the matrices  $R_\theta^{-1}$ ,  $R_\theta$  and the first, second and third derivative matrices of  $R_\theta$ . Hence, from Condition 4 and Lemma 13 used inside Lemma 11, we have  $\sup_{\theta \in \Theta} |(E_{\theta,i,j,\ell})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$  and  $\sup_{\theta \in \Theta} |(F_{\theta,i,j,\ell})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$ . Then, from GCT and Lemma 4 in [28], we have  $\sup_{\theta \in \Theta} \rho_1(E_{\theta,i,j,\ell}) \leq C_{\text{sup}}$  and  $\sup_{\theta \in \Theta} \rho_1(F_{\theta,i,j,\ell}) \leq C_{\text{sup}}$ . Hence, as in the proof of Theorem 3, we can show

$$\sup_{\theta \in \Theta} \left| \frac{\partial L_\theta}{\partial \theta_i \partial \theta_j \partial \theta_\ell} \right| = O_p(1). \quad (37)$$

Also,  $\lambda_1(M_{\theta_0})$  is clearly bounded as  $n \rightarrow \infty$ . From Theorem 2,  $\lambda_1(\Sigma_{\theta_0})$  is bounded as  $n \rightarrow \infty$ . Hence, by considering subsequences along which  $M_{\theta_0}$  and  $\Sigma_{\theta_0}$  converge, and using (34), (35), (36) and (37), we can proceed as in the proof of Proposition D.10 in [7] and show (6).  $\square$

**Proof of Theorem 5.** Let  $\psi \in \mathcal{S}$  be fixed. We have

$$\text{Var}(CV_\psi) = \frac{1}{n} n \text{Var} \left( \frac{1}{n} y^\top C_\psi^{-1} \text{diag}(C_\psi^{-1})^{-2} C_\psi^{-1} y \right).$$

From Lemmas 11, 13 and 14 (that can be trivially adapted by replacing  $\theta$  by  $\psi$ ), as well as Theorem 2, applied with  $A_n = C_\psi^{-1} \text{diag}(C_\psi^{-1})^{-2} C_\psi^{-1}$ , we obtain  $\text{Var}(CV_\psi) \rightarrow 0$  as  $n \rightarrow \infty$ .

For  $i = 1, \dots, p-1$ ,

$$\frac{\partial CV_\psi}{\partial \psi_i} = \frac{2}{n} y^\top A_{\psi,i} y$$

with

$$A_{\psi,i} = C_\psi^{-1} \text{diag}(C_\psi^{-1})^{-2} \left( \text{diag} \left( C_\psi^{-1} \frac{\partial C_\psi}{\partial \psi_i} C_\psi^{-1} \right) \text{diag}(C_\psi^{-1})^{-1} - C_\psi^{-1} \frac{\partial C_\psi}{\partial \psi_i} \right) C_\psi^{-1}.$$

As in the proof of Theorem 3, GCT and Lemma 4 in [28] lead us to

$$\sup_{\psi \in \mathcal{S}} \lambda_1(A_{\psi,i}^\top A_{\psi,i}) \leq C_{\text{sup}},$$

which in turn implies  $\sup_{\psi \in \mathcal{S}} \rho_1(A_{\psi,i}) \leq C_{\text{sup}}$ . It follows that

$$\begin{aligned} \max_{i=1, \dots, p-1} \sup_{\psi \in \mathcal{S}} \left| \frac{\partial CV_\psi}{\partial \psi_i} \right| &\leq \sup_{\psi \in \mathcal{S}} \left| \frac{2}{n} (y^\top A_{\psi,i} y) \right| \\ &\leq C_{\text{sup}} \frac{\|y\|^2}{n} \\ &= O_p(1). \end{aligned}$$

Hence, Theorem 5 can also be proved by proceeding as in the proof of Proposition 3.4 in [7].  $\square$

**Proof of Theorem 6.** From Condition 4 and Lemma 13 used inside Lemma 11, we have for  $i = 1, \dots, p-1$

$$\frac{\partial CV_\psi}{\partial \psi_i} = \frac{2}{n} y^\top A_{\psi,i} y,$$

where  $A_{\psi,i}$  is a  $n \times n$  matrix satisfying  $\sup_{\psi \in \mathcal{S}} |(A_{\psi,i})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$ . As in the proof of Theorem 4, one can check that  $\partial CV_{\psi_0}/\partial \psi_i$  has mean zero for  $i = 1, \dots, p-1$ . Let  $\partial CV_{\psi_0}/\partial \psi$  be the gradient column vector of  $CV_{\psi}$  at  $\psi_0$ . From Theorem 2, with  $\mathcal{L}_{\Gamma, \psi_0, n}$  the distribution of  $\sqrt{n}\partial CV_{\psi_0}/\partial \psi$ , as  $n \rightarrow \infty$ ,

$$d_w(\mathcal{L}_{\Gamma, \psi_0, n}, \mathcal{N}[0, \Gamma_{\psi_0}]) \rightarrow 0. \quad (38)$$

In addition, for  $i \in \{1, \dots, p-1\}$ , the sequence  $(n \text{Var}(\partial CV_{\psi_0}/\partial \psi_i))$  is bounded and thus, the elements of  $\Gamma_{\psi_0}$  are bounded too.

One can check that the mean value of  $\partial CV_{\psi_0}/\partial \psi_i \partial \psi_j$  is  $(N_{\psi_0})_{i,j}$ . Furthermore, from Theorem 2, the variance of  $\partial CV_{\psi_0}/\partial \psi_i \partial \psi_j$  goes to zero as  $n \rightarrow \infty$ . Hence, as  $n \rightarrow \infty$ ,

$$\partial CV_{\psi_0}/\partial \psi_i \partial \psi_j \xrightarrow{p} (N_{\psi_0})_{i,j}. \quad (39)$$

It can be shown, similarly as in the proof of Proposition 3.7 in [7] that

$$\liminf_{n \rightarrow \infty} \lambda_{p-1}(N_{\psi_0}) > 0. \quad (40)$$

Hence, (10) follows. On the other hand, for  $i, j = 1, \dots, p-1$ , we have

$$\frac{\partial CV_{\psi}}{\partial \psi_i \partial \psi_j} = \frac{1}{n} y^\top D_{\psi, i, j} y,$$

where  $D_{\psi, i, j}$  is computed as a sum of products of the matrices  $C_{\psi}^{-1}$ ,  $C_{\psi}$ , the first and second derivative matrices of  $C_{\psi}$  and the diag operator (see e.g., [7]). Hence, from Condition 4 and Lemma 13 used inside Lemma 11, we have  $\sup_{\psi \in \mathcal{S}} |(D_{\psi, i, j})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$ .

Similarly, for  $i, j, \ell \in \{1, \dots, p-1\}$ , we have

$$\frac{\partial CV_{\psi}}{\partial \psi_i \partial \psi_j \partial \psi_\ell} = \frac{1}{n} y^\top E_{\psi, i, j, \ell} y,$$

where  $E_{\psi, i, j, \ell}$  is a sum of products of the matrices  $C_{\psi}^{-1}$ ,  $C_{\psi}$ , the first, second and third derivative matrices of  $C_{\psi}$  and the diag operator. Hence, from Condition 4 and Lemma 13 used inside Lemma 11, we have  $\sup_{\psi \in \mathcal{S}} |(E_{\psi, i, j, \ell})_{a,b}| \leq C_{\text{sup}}/(1 + |s_a - s_b|^{d+C_{\text{inf}}})$ . Then, from GCT and Lemma 4 in [28], we have  $\sup_{\psi \in \mathcal{S}} \rho_1(E_{\psi, i, j, \ell}) \leq C_{\text{sup}}$ . Hence, as in the proof of Theorem 3, we can show, for  $i, j, \ell \in \{1, \dots, p-1\}$ ,

$$\sup_{\psi \in \mathcal{S}} \left| \frac{\partial CV_{\psi}}{\partial \psi_i \partial \psi_j \partial \psi_\ell} \right| = O_p(1). \quad (41)$$

Also,  $\lambda_1(N_{\psi_0})$  is clearly bounded as  $n \rightarrow \infty$ . From Theorem 2,  $\lambda_1(\Gamma_{\psi_0})$  is bounded as  $n \rightarrow \infty$ . Hence, by considering subsequences along which  $N_{\psi_0}$  and  $\Gamma_{\psi_0}$  converge, and using (38), (39), (40) and (41), we can proceed as in the proof of Proposition D.10 in [7] and show (9).  $\square$



**Proof of Lemma 4.** We have, with  $\theta = (\sigma^2, \psi) \in \Theta$  and  $\theta_0 = (\sigma_0^2, \psi_0)$ ,

$$\begin{aligned}
& \frac{1}{n} \sum_{i,j=1}^n (c_{Y,\psi}(s_i - s_j) - c_{Y,\psi_0}(s_i - s_j))^2 \\
&= \frac{1}{n} \sum_{i,j=1}^n \left( \frac{k_{Y,\theta}(s_i - s_j)}{k_{Y,\theta}(0)} - \frac{k_{Y,\theta_0}(s_i - s_j)}{k_{Y,\theta_0}(0)} \right)^2 \\
&\leq \frac{2}{n} \sum_{i,j=1}^n \left( \frac{k_{Y,\theta}(s_i - s_j)}{k_{Y,\theta}(0)} - \frac{k_{Y,\theta_0}(s_i - s_j)}{k_{Y,\theta}(0)} \right)^2 \\
&\quad + \frac{2}{n} \sum_{i,j=1}^n \left( \frac{k_{Y,\theta_0}(s_i - s_j)}{k_{Y,\theta}(0)} - \frac{k_{Y,\theta_0}(s_i - s_j)}{k_{Y,\theta_0}(0)} \right)^2 \\
&\leq \frac{C_{\sup}}{n} \sum_{i,j=1}^n (k_{Y,\theta}(s_i - s_j) - k_{Y,\theta_0}(s_i - s_j))^2 + \left( \frac{1}{k_{Y,\theta}(0)} - \frac{1}{k_{Y,\theta_0}(0)} \right)^2 C_{\sup},
\end{aligned} \tag{42}$$

where the second  $C_{\sup}$  comes from Lemma 12 and from the classical control of the square Frobenius norm by  $n$  times the largest square eigenvalue, for  $n \times n$  symmetric matrices. If Condition 9 holds, then for all  $\chi > 0$ ,

$$\liminf_{n \rightarrow \infty} \inf_{\|\psi - \psi_0\| \geq \chi} \frac{1}{n} \sum_{i,j=1}^n (c_{Y,\psi}(s_i - s_j) - c_{Y,\psi_0}(s_i - s_j))^2 > 0.$$

Consider a sequence  $\theta_n = (\sigma_n^2, \psi_n) \in \Theta$  such that  $\|\theta_n - \theta_0\| \geq \chi$ . If we can extract a subsequence  $n_m$  such that  $\liminf_{m \rightarrow \infty} (\sigma_{n_m}^2 - \sigma_0^2)^2 > 0$ , then clearly

$$\liminf_{m \rightarrow \infty} \frac{1}{n_m} \sum_{i,j=1}^{n_m} (k_{Y,\theta_{n_m}}(s_i - s_j) - k_{Y,\theta_0}(s_i - s_j))^2 > 0$$

by considering the diagonal terms in the above double sum. If we can not extract such a subsequence, then we can extract a subsequence  $n_m$  such that  $\|\psi_{n_m} - \psi_0\| \geq \chi/2$  and  $\sigma_{n_m}^2 \rightarrow \sigma_0^2$  as  $m \rightarrow \infty$ . Along this subsequence

$$\liminf_{m \rightarrow \infty} \frac{1}{n_m} \sum_{i,j=1}^{n_m} (k_{Y,\theta_{n_m}}(s_i - s_j) - k_{Y,\theta_0}(s_i - s_j))^2 > 0$$

from (42). Hence, Condition 7 holds.

Let us now assume that Condition 8 does not hold. We have, with  $\theta = (\sigma^2, \psi) \in \Theta$ , with  $\theta_0 = (\sigma_0^2, \psi_0)$  and with  $(\beta_1, \dots, \beta_p) = (\beta_1, \chi_1, \dots, \chi_{p-1})$ , where  $\beta_1 \in \mathbb{R}$  is arbitrary,

$$\frac{1}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^{p-1} \chi_\ell \frac{\partial}{\partial \psi_\ell} c_{Y,\psi_0}(s_i - s_j) \right)^2$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^p \beta_\ell \frac{\partial}{\partial \theta_\ell} \left( \frac{k_{Y,\theta_0}(s_i - s_j)}{k_{Y,\theta_0}(0)} \right) \right)^2 \\
&= \frac{1}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^p \beta_\ell \frac{\frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(s_i - s_j)}{k_{Y,\theta_0}(0)} - \sum_{\ell=1}^p \beta_\ell \frac{k_{Y,\theta_0}(s_i - s_j) \frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(0)}{k_{Y,\theta_0}(0)^2} \right)^2 \\
&\leq \frac{2}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^p \beta_\ell \frac{\frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(s_i - s_j)}{k_{Y,\theta_0}(0)} \right)^2 \\
&\quad + \frac{2}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^p \beta_\ell \frac{k_{Y,\theta_0}(s_i - s_j) \frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(0)}{k_{Y,\theta_0}(0)^2} \right)^2 \\
&\leq \frac{C_{\sup}}{n} \sum_{i,j=1}^n \left( \sum_{\ell=1}^p \beta_\ell \frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(s_i - s_j) \right)^2 + C_{\sup} \left( \sum_{\ell=1}^p \beta_\ell \frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(0) \right)^2,
\end{aligned} \tag{43}$$

where the second  $C_{\sup}$  comes from Lemma 12 and from the classical control of the square Frobenius norm by  $n$  times the largest square eigenvalue, for  $n \times n$  symmetric matrices. If Condition 8 does not hold, there exists  $(\beta_1^*, \dots, \beta_p^*) \neq (0, \dots, 0)$  and a subsequence  $n_m$  such that

$$\frac{1}{n_m} \sum_{i,j=1}^{n_m} \left( \sum_{\ell=1}^p \beta_\ell^* \frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(s_i - s_j) \right)^2 \xrightarrow{m \rightarrow \infty} 0$$

and thus, considering the diagonal elements in the double sum above,

$$\beta_1^* = \sum_{\ell=1}^p \beta_\ell^* \frac{\partial}{\partial \theta_\ell} k_{Y,\theta_0}(0) = 0.$$

Hence, from (43), letting  $(\beta_1^*, \dots, \beta_p^*) = (0, \chi_1^*, \dots, \chi_{p-1}^*)$ , we have

$$\frac{1}{n_m} \sum_{i,j=1}^{n_m} \left( \sum_{\ell=1}^{p-1} \chi_\ell^* \frac{\partial}{\partial \psi_\ell} c_{Y,\psi_0}(s_i - s_j) \right)^2 \rightarrow 0$$

and thus Condition 10 does not hold.  $\square$

**Proof of Theorem 7.** Let  $\lambda$  and  $\gamma$  be two column vectors in  $\mathbb{R}^p$  and  $\mathbb{R}^{p-1}$ . Let also

$$W_n = \lambda^\top (\hat{\theta}_{\text{ML}} - \theta_0) + \gamma^\top (\hat{\psi}_{\text{CV}} - \psi_0),$$

for  $n \in \mathbb{N}$ .

From the proofs of Theorems 4 and 6 (see also the proof of Proposition D.10 in [7] that is referred to there), we know

$$\sqrt{n}(\hat{\theta}_{\text{ML}} - \theta_0) = \sqrt{n} M_{\theta_0}^{-1} \frac{\partial}{\partial \theta} L_{\theta_0} + o_p(1)$$

and

$$\sqrt{n}(\hat{\psi}_{\text{CV}} - \psi_0) = \sqrt{n}N_{\psi_0}^{-1} \frac{\partial}{\partial \psi} CV_{\psi_0} + o_p(1).$$

Also, from Condition 4 and Lemma 13 used inside Lemma 11, we have for  $i = 1, \dots, p$ ,

$$\partial L_{\theta_0} / \partial \theta_i = \frac{1}{n} (y^\top A_{\theta_0, i} y) + c_{\theta_0},$$

where  $c_{\theta_0} \in \mathbb{R}$  is deterministic and, for  $i = 1, \dots, p-1$ ,

$$\partial CV_{\psi_0} / \partial \psi_i = \frac{1}{n} (y^\top B_{\psi_0, i} y),$$

where  $A_{\theta, i}$  is a  $n \times n$  symmetric matrix satisfying  $\sup_{\theta \in \Theta} |(A_{\theta, i})_{a, b}| \leq C_{\text{sup}} / (1 + |s_a - s_b|^{d+C_{\text{inf}}})$  and  $B_{\psi, i}$  is a  $n \times n$  matrix satisfying  $\sup_{\psi \in \mathcal{S}} |(B_{\psi, i})_{a, b}| \leq C_{\text{sup}} / (1 + |s_a - s_b|^{d+C_{\text{inf}}})$ . As in the proofs of Theorems 4 and 6, one can check that  $\partial L_{\theta_0} / \partial \theta_i$  has mean zero for  $i = 1, \dots, p$  and  $\partial CV_{\psi_0} / \partial \psi_i$  has mean zero for  $i = 1, \dots, p-1$ . Thus, we can rewrite

$$W_n = J_n - \mathbb{E}[J_n] + o_p(n^{-1/2})$$

with

$$J_n = \frac{1}{n} y^\top \left( \sum_{i=1}^p (\lambda^\top M_{\theta_0}^{-1})_i A_{\theta_0, i} + \sum_{i=1}^{p-1} (\gamma^\top N_{\psi_0}^{-1})_i B_{\psi_0, i} \right) y.$$

As the vectors  $\lambda$  and  $\gamma$  as well as the matrices  $M_{\theta_0}^{-1}$  and  $N_{\psi_0}^{-1}$  are fixed, the bound

$$\left| \left( \sum_{i=1}^p (\lambda^\top M_{\theta_0}^{-1})_i A_{\theta_0, i} + \sum_{i=1}^{p-1} (\gamma^\top N_{\psi_0}^{-1})_i B_{\psi_0, i} \right)_{k, l} \right| \leq \frac{C_{\text{sup}}}{1 + |s_k - s_l|^{d+C_{\text{inf}}}}$$

holds for all  $k, l \in \{1, \dots, n\}$ .

Let then  $\mathcal{L}_{J, \theta_0, n}$  be the distribution of  $n^{1/2}(J_n - \mathbb{E}[J_n])$ . Then, from Theorem 2, as  $n \rightarrow \infty$ ,

$$d_w(\mathcal{L}_{J, \theta_0, n}, \mathcal{N}[0, n \text{Var}(J_n)]) \rightarrow 0.$$

The variance can be written as

$$\begin{aligned} n \text{Var}(J_n) &= \sum_{i=1}^p \sum_{j=1}^p (\lambda^\top M_{\theta_0}^{-1})_i (\lambda^\top M_{\theta_0}^{-1})_j (\Sigma_{\theta_0})_{i, j} \\ &\quad + \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} (\gamma^\top N_{\psi_0}^{-1})_i (\gamma^\top N_{\psi_0}^{-1})_j (\Gamma_{\psi_0})_{i, j} \\ &\quad + 2 \sum_{i=1}^p \sum_{j=1}^{p-1} (\lambda^\top M_{\theta_0}^{-1})_i (\gamma^\top N_{\psi_0}^{-1})_j (\Omega_{\theta_0})_{i, j} \end{aligned}$$

with

$$\Omega_{i,j} = \text{Cov} \left( \sqrt{n} \frac{\partial}{\partial \theta_i} L_{\theta_0}, \sqrt{n} \frac{\partial}{\partial \psi_j} CV_{\psi_0} \right).$$

Hence, by applying product by blocks we get the matrix form expression

$$n \text{Var}(J_n) = (\lambda^\top, \gamma^\top) D_{\theta_0}^{-1} \Psi_{\theta_0} D_{\theta_0}^{-1} \begin{pmatrix} \lambda \\ \gamma \end{pmatrix}.$$

We conclude the proof by applying the Wald Theorem.  $\square$

**Proof of Proposition 1.** The proofs of the new versions of the theorems are direct extensions of the proofs of these theorems above. In particular, Condition 11 implies (17) in Lemma 12. Furthermore, Theorem 2 can be shown to hold. For instance, one can write, for  $s \in \mathbb{R}^d$ ,  $Y(s) = T(Z(s)) + \dot{T}(\zeta(s)) = \dot{T}(\dot{Z}(s))$ , where  $\dot{Z} = (Z, \zeta)$  is a bivariate zero-mean Gaussian process and where  $\dot{T} : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies  $|\dot{T}(x)| \leq C_{\text{sup}} \exp(C_{\text{sup}} \|x\|)$  and  $|\partial/\partial x_i \dot{T}(x)| \leq C_{\text{sup}} \exp(C_{\text{sup}} \|x\|)$  for  $x \in \mathbb{R}^2$  and  $i = 1, 2$ . We recall that  $Z$  and  $\zeta$  are independent and  $\text{Cov}(\zeta(u), \zeta(v)) = \kappa 1_{u=v}$ , for  $u, v \in \mathbb{R}^d$ , with a constant  $0 < \kappa < \infty$ .

Then one can check that the proof of Theorem 2 can be repeated, almost identically, by simply replacing  $Z$  by  $\dot{Z}$  and  $T$  by  $\dot{T}$ . Finally, we remark that for (non-transformed) Gaussian processes with a nugget effect, [11, 13] address the asymptotic properties of maximum likelihood. They observe that the arguments for Gaussian processes without nugget effect in [7] can be extended directly to Gaussian processes with nugget effect.  $\square$

**Proof of Corollary 3.** When  $Y$  has constant mean  $\nu_0$  and  $Z$  has constant mean  $\mu_0$ , then we can apply Theorem 2 to the Gaussian process  $\dot{Z} = Z - \mu_0$ , to the transformation  $\dot{T}$  defined by  $\dot{T}(x) = T(x + \mu_0)$  for  $x \in \mathbb{R}$  and to the transformed process  $\dot{Y} = \dot{T}(\dot{Z}) - \nu_0$ . This yields a central limit theorem for quadratic forms based on the centered observation vector  $y - \nu_0$ , which concludes the proof.

Note that when  $T$  satisfies Condition 3, then  $\dot{T}$  also satisfies Condition 3.  $\square$

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